ORIGINAL ARTICLE

Stability Analysis of Delayed Cournot Model in the sense of Lyapunov

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Abstract:

In this paper the dynamic equation of Duopoly production model in certain two firms is considered. The existence of best responses that can maximize profit, and stability conditions are analyzed when one of the players or both of them have delayed information and/or delayed actions. A system of nonlinear delayed differential equations and Lyapunov method of nonlinear stability analysis are employed. It is ascertained that, in the case of equal and fixed information delay in both the firms, the delay causes oscillatory process in the system and does not affect the qualitative behavior of the solution (no effect on the stability of the Nash equilibrium point), but only changes the transition process. On the other hand, when one of the firms has implementation delay and the rival player makes decision without delay, it leads to instability of the dynamic system at least locally. The same result is obtained when one of the firms has implementation and the other information delay. Numerical simulation using MATLAB2012a is used to demonstrate the applicability and accuracy of the results.

KEYWORDS: *Delay Differential Equations, Nonlinear dynamic system, Stability, Lyapunov method, method of linearization, Nash equilibrium, duopoly model.*

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INTRODUCTION

The study of dynamic systems in the form of $\dot{x} = f(x,t),$ where x and f are vectors in $R^n, n = 1, 2, \ldots,$has been subject of nonlinear dynamic research in Economics (Zhang,2005;Peters,1994) with the goal of getting better understanding of phenomena such as economic growth, prediction and control their behavior. A wide range of applications of nonlinear models like Duopoly models have been devised and applied in Economics.

Duopoly is a market model in which there are two (Bischi & Namzada,2000; Elsadany,2010) different rules–*quantity-setting* rule which is referred to as the Cournot model after the economist Cournot, and the *price-setting* rule called the Bertrand model after the economist Bertrand. In this paper the mathematical model of quantity-setting rule or game (Cournot model) is considered and its stability relative its Nash equilibrium point is investigated. It is assumed that the firms are identical in their technology and in the goods they are producing.

Moreover, the assumption that the equilibrium of the two firms exist on the intersection of the two firms "reaction functions" is considered. Hence, the investigation starts from finding these two reaction functions. The impact of time delay on the stability of the equilibrium points of the competing firms, described by delay differential equation is investigated.

Thus, the purpose of this paper is to analyze the stability (in the sense of Lyapunov) of Cournot model based on different delays such as information and action delays of the competitors.

The Model and Statement of the Problem

In Cournot model, each firm decides the amount of production, considering the amount by its competitors as a given constant. In this game, the two duopolies, simultaneously and independently, decide on the amount of production. Consequently, let firm 1 chooses x_1 and firm 2 chooses x_2 amount of production. Then the market decides the price of the product, p , through the aggregate demand (Kopel,1996) curve given by: $p = a - b(x_1 + x_2),$ where $a > 0, b > 0,$ are parameters representing respectively, the maximum selling price and the opposite of the slope of the inverse demand function. It is assumed that $x_1 + x_2$ is the total production and both firms have the same cost function given by $C(x_i) = c_i x_i, i = 1, 2$, where x_i is the quantity produced and c_i is a constant marginal cost.

It is known that the profit of a firm is given by

$$\pi_i(x_1, x_2) := p x_i - c_i x_i = (p - c_i) x_i, i = 1, 2 .$$

That is

$$\begin{cases} \pi_1 = (a - c_1 - b x_2) x_1 - b x_1^2 & (1a) \\ \pi_2 = (a - c_2 - b x_1) x_2 - b x_2^2 & (1b) \end{cases}$$

Note that, eqs. (1a) and (1b) are functions of both x_1 and x_2. Equation(1a) is linear in x_2 and as x_2 increases the profits of firm 1 decrease – since the coefficient on x_2 is negative. It is second degree in x_1 with a negative coefficient on x_1. So as x_1 increases then first the profits of firm 1 increases and then decrease.The same condition applies to eq.(1b).

The dynamics in which the firms adjust their level of production depends on the gradient of marginal profit and (Agiza & Elsadany,2003;Bischi & Namzada,2000) is given by:

$$x_i(t+1) - x_i(t) = \mu_i(x_i)\frac{\partial \pi_i}{\partial x_i} \quad , \quad i = 1,2 \quad , \tag{2}$$

where $\mu_i(x_i)$ is a positive function which gives the extent of production variation of the i^{th} firm following a given profit signal π_i (adjustment function). In this paper for simplicity we assume $\mu_i(x_i) = x_i$.

Now using eqs. (1), (2), the assumptions and replacing $\dot{x}(t) = x(t+1) - x(t)$ we have dynamic system of the form:

$$\begin{cases} \dot{x}_1(t) = bx_1(t)\left[b_1 - 2x_1(t) - x_2(t)\right] \\ \dot{x}_2(t) = bx_2(t)\left[b_2 - x_1(t) - 2x_2(t)\right] \end{cases}, \tag{3}$$

where, $b_i = \dfrac{a - c_i}{b}$.

Taking into account an information delay on the dynamics of the processes of interaction of competing firms, one possible way of describing eq.(3) is by delay differential equation system given as

$$\begin{cases} \dot{x}_1(t) = bx_1(t)\left[b_1 - 2x_1(t) - x_2(t-\tau)\right] \\ \dot{x}_2(t) = bx_2(t)\left[b_2 - x_1(t-\tau) - 2x_2(t)\right] \end{cases}, \tag{4}$$

where τ – is a positive number and is a delay parameter.

Based on eq.(4) the first part of this paper investigated the stability of its Nash equilibrium point. Moreover, other models obtained by modifying eq. (4) with different delay conditions are developed and their stability investigated at the Nash equilibrium point.

Nash Equilibrium Point

An equilibrium point or stationary point (Bischi & Namzada,2000; Smith, 2011) of the dynamic duopoly model is defined as a nonnegative fixed point of eq.(3). In eq. (3) assuming that each duopolies is trying to increase profit, we can have at most four equilibrium points: $E_0 = (0,0), E_1 = \left(\frac{b_1}{2}, 0\right)$, $E_3 = \left(0, \frac{b_2}{2}\right)$ provided that $a > c_1$, $a > c_2$ .These are called *boundary equilibria*. Now let $\pi_1 = x_1 b\left[b_1 - 2x_1 - x_2\right]$ and $\pi_2 = x_2 b\left[b_2 - x_1 - 2x_2\right]$.Then given some output x_2 for firm1 and x_1 for firm2, the best response that maximizes profit of firm 1 and firm 2 are respectively given as:

$$\frac{\partial \pi_1}{\partial x_1} = 0 \Rightarrow x_1 = \frac{x_2 - b_1}{2}, \quad \text{and} \quad \frac{\partial \pi_2}{\partial x_2} = 0 \Rightarrow x_2 = \frac{x_1 - b_2}{2}.$$

It then follows that, the system of delay differential equations given by eq.(3) has a unique positive **Nash equilibrium point** $x^* = \left(x_1^*, x_2^*\right)$ given by:

$$x_1^* = \frac{2b_1 - b_2}{3} > 0, \; x_2^* = \frac{2b_2 - b_1}{3} > 0, \qquad (4a)$$

under the conditions $2b_1 > b_2, 2b_2 > b_1$.

Stability Analysis

Stability analysis is made for Nash equilibrium points and different delay parameters accompanied by Numerical simulations.

Global Stability Analysis of Time Delay of one Firm over the other.

The dynamics of the processes of interaction of competing firms for time delay of one firm over the other can be described by the system of delay differential equation in eq. (4). Assuming $\overline{x}_1 = x_1 - x_1^*, \overline{x}_2 = x_2 - x_2^*$ in eq.(4) leads to:

$$\begin{cases} \dot{x}_1(t) = b\left(x_1^* + \overline{x}_1(t)\right)\left[-2\overline{x}_1(t) - \overline{x}_2(t-\tau)\right] \\ \dot{x}_2(t) = b\left(x_2^* + \overline{x}_2(t)\right)\left[-\overline{x}_1(t-\tau) - 2\overline{x}_2(t)\right] \end{cases}. \qquad (5)$$

In order to prove the globally uniformly asymptotic stability of the equilibrium position $x^* = \left(x_1^*, x_2^*\right)$ for system of eq.(4) we prove the globally uniformly asymptotical (Xiaoxin, Liqiu, & Yu , 2007) stability of the equilibrium solution of system (5) in the region $\{\overline{x}_1 > -x_1^*, \overline{x}_2 > -x_2^*\}$. For this purpose let us consider a Lyapunov function candidate given by

$$V(\phi_1,\phi_2)=16\left(\phi_1(0)-x_1^* ln\frac{\phi_1(0)+x_1^*}{x_1^*}\right)+16\left(\phi_2(0)-x_2^* ln\frac{\phi_2(0)+x_2^*}{x_2^*}\right)+$$
$$\int_{-\tau}^{0}\phi_2^2(\theta)d\theta+4\int_{-\tau}^{0}\phi_1^2(\theta)d\theta,$$

where ϕ_1,ϕ_2 are continuous functions in $[-\tau,0]$.

The derivative of the candidate function, $\dot{V}$, in the direction of system of eq.(5) leads to

$$\dot{V}(\phi_1,\phi_2)=16\left(\dot{\phi}_1(0)-\frac{\dot{\phi}_1(0)-x_1^*}{\phi_1(0)+x_1^*}\right)+16\left(\dot{\phi}_2(0)-\frac{\dot{\phi}_2(0)x_2^*}{\phi_2(0)+x_2^*}\right)+$$
$$\left(\phi_2^2(0)-\phi_2^2(-\tau)\right)+4\left(\phi_1^2(0)-\phi_1^2(-\tau)\right),$$
$$\leq-\left(-2\phi_1(0)-\phi_2(-\tau)\right)^2-4\left(-\phi_1(-\tau)-2\phi_1(0)\right)^2.$$

Therefore, the solution $\left(x_1(t),x_2(t)\right)\in\left\{\dot{V}\left(x(t)\right)=0\right\}$ whenever

$$\left\{-2x_1(t)-x_2(t-\tau)=0,\qquad -x_1(t-\tau)-2x_2(t)=0\right\}.$$

In order to prove the global stability, we need to show that system (5) contains only trivial solution (Liao, Wang & Yu, 2007) in the maximal invariant set M of the subset of solution set.

Assume that $\mathrm{x}(t)=\left(x_1(t),x_2(t)\right)$ is an arbitrary solution contained in the largest invariant set M . That is the solutions must satisfy

$$-2x_1(t)-x_2(t-\tau)=0 \text{ and } -x_1(t-\tau)-2x_2(t)=0,\qquad for\qquad all\qquad t\geq t_0.$$

We want to show that $\mathrm{x}(t)=(0,0)$. System of eqs. (3) , (5) leads to

$x_1=0,x_2=0,$ Indeed, if $\begin{cases}\dot{x}_1(t)=0\\ \dot{x}_2(t)=0\end{cases}$,then $\begin{cases}x_1(t)=const=k_1\\ x_2(t)=const=k_2\end{cases}$, and

hence, $\begin{cases}-2k_1-k_2=0\\ -k_1-2k_2=0\end{cases}$.

The linear system related to k_1,k_2 have only trivial solution since the determinate is non-zero. Thus, the solution $x^*=\left(\frac{2b_1-b_2}{3},\frac{2b_2-b_1}{3}\right)$ is globally uniformly asymptotically stable for system of eq.(5).

Numerical Simulation I

Numerical simulation based on different delay parameters is conducted using MATLAB.

a). Let $x^* = \left(\frac{1}{3}, \frac{1}{3}\right)$, $b = 1$, then eq.(5) becomes:

$$\begin{cases} \dot{\overline{x}}_1(t) = \left(\frac{1}{3} + \overline{x}_1(t)\right)\left[-2\,\overline{x}_1(t) - \overline{x}_2(t-\tau)\right] \\ \dot{\overline{x}}_2(t) = \left(\frac{1}{3} + \overline{x}_2(t)\right)\left[-\overline{x}_1(t-\tau) - 2\,\overline{x}_2(t)\right] \end{cases}$$

Simulation result of this system for $x^* = \left(\frac{1}{3}, \frac{1}{3}\right)$, and different delay parameters τ, $(\tau = 0.9, \tau = 9.9, \tau = 20)$ is shown below in fig1.

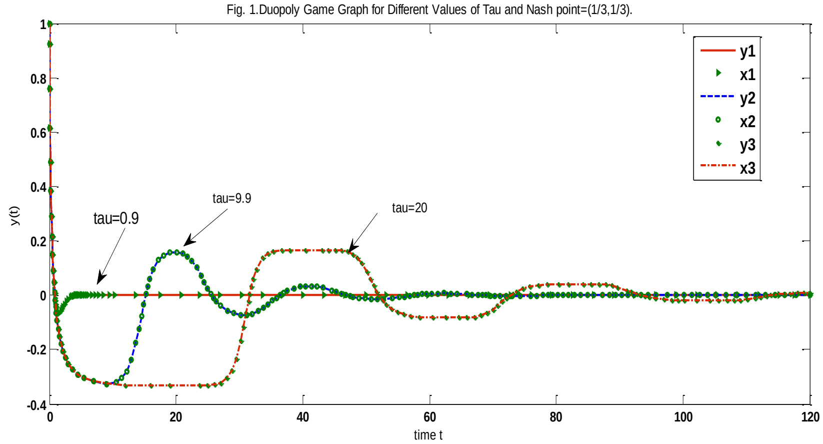


Fig. 1.Duopoly Game Graph for Different Values of Tau and Nash point=(1/3,1/3).

b) .Let $x^* = \left(\frac{20}{3}, \frac{20}{3}\right)$, $b = 1$, then the simulation result is shown in fig.2 below.

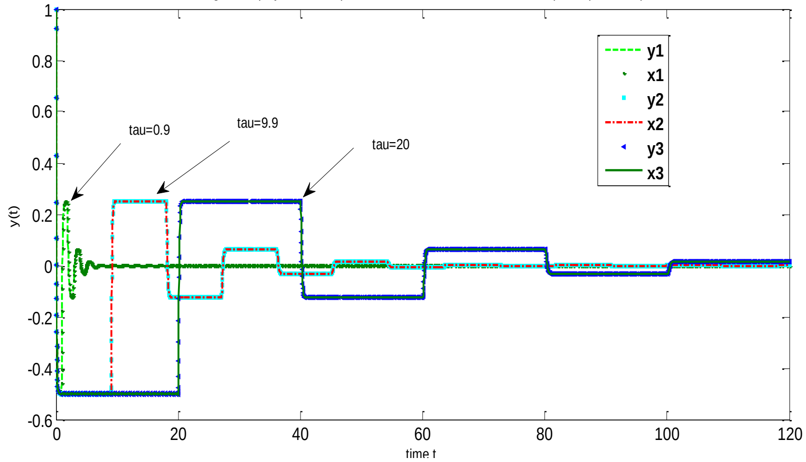


Fig.2. Duopoly Game Graph for Different Values of Tau and Nash point=(20/3,20/3)

In conclusion, one can infer from the above figures (fig1and 2) that, the effect of the delay on the model causes oscillatory process in the system. The delay does not affect the qualitative behavior of the solutions, but only changes the transition time process. In other words it delays stability as the delay parameter, τ , increases

Stability Analysis of Own Time Delay of one of the Firms and no delay of the second Firm.

In this section we examine *local effect* caused by own implementation delay of one of the firms and the other being with no delay, on the stability of the Nash equilibrium point.

Equation (5) in general can be lineaarized and be written in the form:

$$\dot{X} = AX,$$

where

$$A = b\begin{pmatrix} \frac{4b_2 - 11b_1}{3} & \frac{2b_2 - 4b_1}{3} \\ \frac{2b_1 - 4b_2}{3} & \frac{4b_1 - 11b_2}{3} \end{pmatrix}, X = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, b_1 = \frac{a - c_1}{b}, b_2 = \frac{a - c_2}{b}.$$

The new linearized form of equation (3) is now given as:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = b\begin{pmatrix} \frac{4b_2 - 11b_1}{3} & \frac{2b_2 - 4b_1}{3} \\ \frac{2b_1 - 4b_2}{3} & \frac{4b_1 - 11b_2}{3} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

A dynamic system, using the linearized form of eq.(3), representing own implementation delay of one of the firms and the other being with no delay can be described as

$$\begin{cases} \dot{\overline{x}}_1(t) = b\left(\dfrac{4b_2 - 11b_1}{3}\right)\overline{x}_1(t-\tau_1) + b\left(\dfrac{2b_2 - 4b_1}{3}\right)\overline{x}_2(t) \\ \dot{\overline{x}}_2(t) = b\left(\dfrac{2b_1 - 4b_2}{3}\right)\overline{x}_1(t) + b\left(\dfrac{4b_1 - 11b_2}{3}\right)\overline{x}_2(t) \end{cases} . \qquad (7)$$

Substituting the exponential forms $\overline{x}_1(t) = ue^{\lambda t}$ and $\overline{x}_2(t) = ve^{\lambda t}$ into eq.(7) leads to

$$\begin{pmatrix} \dot{\overline{x}}_1(t) \\ \dot{\overline{x}}_2(t) \end{pmatrix} = \begin{pmatrix} \lambda + \alpha_1 e^{-\lambda\tau_1} & \alpha_2 \\ \alpha_3 & \lambda + \alpha_4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \qquad (8)$$

where, $\alpha_1 = \dfrac{11b_1 - 4b_2}{3}, \alpha_2 = \dfrac{4b_1 - 2b_2}{3}, \alpha_3 = \dfrac{4b_2 - 2b_1}{3} . \alpha_4 = \dfrac{11b_2 - 4b_1}{3}$, $\tau_1 > 0$ is implementation delay of firm 1.

Non-trivial solution exists if and only if

$$\lambda^2 + \alpha_4\lambda + \alpha_1(\lambda + \alpha_4)e^{-\lambda\tau_1} - \alpha_3\alpha_2 = 0. \qquad (9)$$

Before we are going to look for the roots, let us consider the case where $\tau_1 = 0$ in eq.(9). Now consider

$$\lambda^2 + (\alpha_1 + \alpha_4)\lambda + \alpha_1\alpha_4 - \alpha_3\alpha_2 = 0. \qquad (9^*)$$

In eq. (9^*), $\alpha_1 + \alpha_4 = \dfrac{7(b_1 + b_2)}{3} > 0$ for otherwise it contradicts the assumptions made in (4a).
Moreover, $\alpha_1\alpha_4 - \alpha_3\alpha_2 > 0$, since $\alpha_i > 0, i = 1,2,3,4$ and $\alpha_1 > \alpha_2, \alpha_4 > \alpha_3$. As a result all the coefficients of λ in eq.(9*) are positive and hence all the eigenvalues of eq.(8) have a negative real part. It can then be concluded that, the nonlinear system represented by eq. (3) with no delay has a *locally asymptotically stable equilibrium* point.

The next question is to look for a cutoff value (if any) of the delay for which this stability of the equilibrium point is lost. To examine whether such a treshhold value exists or not put $\lambda = i\omega,\ \omega > 0$ in eq.(9) and obtain

$$-\omega^2 + i\alpha_4\omega + \alpha_1(i\omega + \alpha_4)(cos\tau_1\omega - isin\tau_1\omega) - \alpha_3\alpha_2 = 0. \qquad (9a)$$

Separating the real and imaginary part of eq. (9a) leads to

$$\begin{cases}\alpha_1\left(\alpha_4 cos\tau_1\omega+\omega sin\tau_1\omega\right)=\omega^2+\alpha_3\alpha_2\\ \alpha_1\left(\omega cos\tau_1\omega-\alpha_4 sin\tau_1\omega\right)=-\alpha_4\omega\end{cases}. \tag{10}$$

Squaring each part in eq.(10) and adding we obtain:

$$\omega^4+\left(\alpha_4^2-\alpha_1^2+2\alpha_3\alpha_4\right)\omega^2+\left(\alpha_3\alpha_2\right)^2-\left(\alpha_1\alpha_4\right)^2=0. \tag{11}$$

From eq.(11) the root is obtained to be:

$$\omega_0=:\sqrt{\frac{-\left(\alpha_4^2-\alpha_1^2+2\alpha_3\alpha_4\right)+\sqrt{\left(\alpha_4^2-\alpha_1^2+2\alpha_3\alpha_4\right)^2-4\left[\left(\alpha_3\alpha_2\right)^2-\left(\alpha_1\alpha_4\right)^2\right]}}{2}}>0. \tag{11a}$$

Note that, $\left(\alpha_3\alpha_2\right)^2-\left(\alpha_1\alpha_4\right)^2<0$,

and $\left(\alpha_4^2-\alpha_1^2+2\alpha_3\alpha_4\right)^2-4\left[\left(\alpha_3\alpha_2\right)^2-\left(\alpha_1\alpha_4\right)^2\right]>0,$ which implies that eq.(11) has only positive real roots. Hence, ω_0 is the only positive root of eq. (11). Consequently, eq.(9) has only purely imaginary roots, $\lambda=\pm i\omega_0$.

To find τ_1 solve for $cos\tau_1\omega$ from eq.(10) and obtain:

$$\tau_1^n=\frac{1}{\omega_0}cos^{-1}\left(\frac{\alpha_2\alpha_3\alpha_4}{\alpha_1\left(\omega_0+\alpha_4^2\right)}\right)+\frac{2n\pi}{\omega_0}, n=0,1,2,\ldots \quad . \tag{12}$$

Since the real part of all the roots of the characteristic equation of eq.(9) are zero , we can say that:

$$\tau_1=\frac{1}{\omega_0}cos^{-1}\left(\frac{\alpha_2\alpha_3\alpha_4}{\alpha_1\left(\omega_0+\alpha_4^2\right)}\right),$$

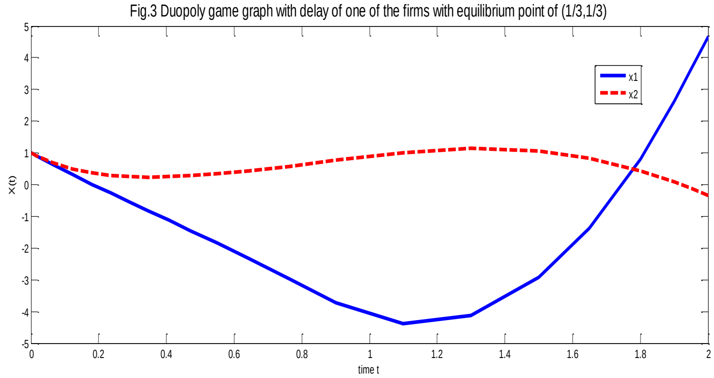
is the smallest (among τ_1^n) cutoff value at which stability of the equilibrium point is lost. For all values $\tau_1^n\geq\tau_1$ the Nash equilibrium point of delay differential equation in eq. (7) is unstable. A routine bifurcation analysis shows that the stability can't be regained in later time t. The numerical simulation result in figures 3 and 4 also shows that the stability can't be regained.

Numerical Simulation II

a). Let $x^*=\left(\frac{1}{3},\frac{1}{3}\right)$, $b=1$, $x_1^*=\frac{2b_1-b_2}{3}>0$, $x_2^*=\frac{2b_2-b_1}{3}>0$, Then $b_1=b_2=2$ then eq.(7) becomes:

$$\begin{cases} \dot{\overline{x}}_1(t) = \frac{-14}{3}\overline{x}_1(t-\tau_1) + \frac{-4}{3}\overline{x}_2(t) \\ \dot{\overline{x}}_2 = \frac{-4}{3}\overline{x}_1(t) + \frac{-14}{3}\overline{x}_2(t) \end{cases}.$$

The simulation result for this section is shown in fig.3 below for an equilibrium point $x^* = (\frac{1}{3}, \frac{1}{3})$.



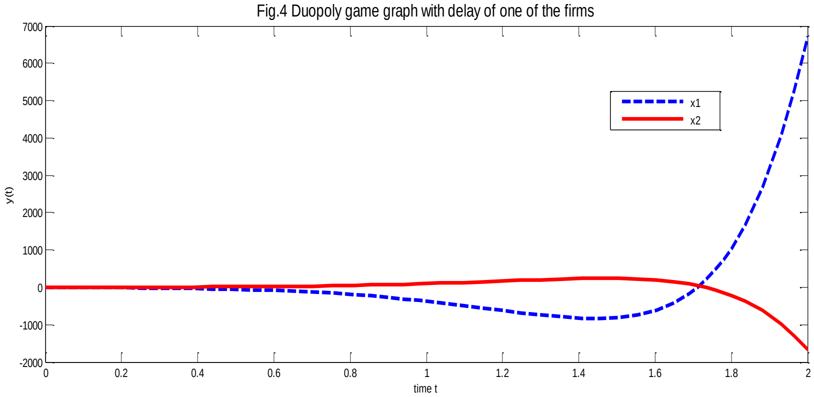
It can be inferred from fig. 3 that the stability of the equilibrium is lost after some time from the start and increasing the simulation times howed, there is no chance of regaining the stability lost.

b). Let $x^* = \left(\frac{20}{3}, \frac{20}{3}\right)$, $b = 1, x_1^* = \frac{2b_1 - b_2}{3} > 0,\ x_2^* = \frac{2b_2 - b_1}{3} > 0,$ Then $b_1 = b_2 = 2$ 0.Then eq.(7) reduces to:

$$\begin{cases} \dot{\overline{x}}_1(t) = \frac{-140}{3}\overline{x}_1(t) - \frac{-40}{3}\overline{x}_2(t-\tau) \\ \dot{\overline{x}}_2(t) = \frac{-40}{3}\overline{x}_1(t-\tau) - \frac{140}{3}\overline{x}_2(t) \end{cases}$$

The simulation result for this section is shown in fig.4 below for an equilibrium point of $x^* = (\frac{20}{3}, \frac{20}{3})$.

Fig.4 Duopoly game graph with delay of one of the firms



It can be inferred from the fig. 4 that, the stability of the equilibrium is lost after some time from the early start. Demonstartion by increasing the simulation time shows that the stability can't be regained in any future time.

Stability Analysis of Own Delay of one the Firms and Information Delay of the second Firm

A dynamic system based on eq.(3) representing own delay of one of the firm and information delay of the other can be described as:

$$\begin{cases} \dot{\bar{x}}_1(t) = b\left(\frac{4b_2-11b_1}{3}\right)\bar{x}_1(t) + b\left(\frac{2b_2-4b_1}{3}\right)\bar{x}_2(t-\tau_1) \\ \dot{\bar{x}}_2(t) = b\left(\frac{2b_1-4b_2}{3}\right)\bar{x}_1(t) + b\left(\frac{4b_1-11b_2}{3}\right)\bar{x}_2(t-\tau_2) \end{cases}. \quad (13)$$

In the same way we did in section 4.2, substituting the exponential forms

$\dot{x}_1(t) = ue^{\lambda t}$ and $\dot{x}_2(t) = ve^{\lambda t}$ into the linearized equation (13) we obtain:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \lambda+\alpha_1 & \alpha_2 e^{-\lambda\tau_1} \\ \alpha_3 & \lambda+\alpha_4 e^{-\lambda\tau_2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad (14)$$

where

$\alpha_1 = \frac{11b_1-4b_2}{3}, \alpha_2 = \frac{4b_1-2b_2}{3}, \alpha_3 = \frac{4b_2-2b_1}{3}. \alpha_4 = \frac{11b_2-4b_1}{3}$, $\tau_1, \tau_2 > 0$ and are information and implementation delays of firm 1 and firm 2 respectively .Non-trivial solution in eq.(14) exists if and only if

$$\lambda^2 + \lambda\left(\alpha_4 e^{-\lambda\tau_2} + \alpha_1\right) + \alpha_1\alpha_4 e^{-\lambda\tau_2} - \alpha_3\alpha_2 e^{-\lambda\tau_1} = 0.$$

Putting $\tau_1 = \tau_2 = \tau$,for simplicity, we obtain:

$$\lambda^2 + \lambda\left(\alpha_4 + \alpha_1\right) + (\alpha_1\alpha_4 - \alpha_3\alpha_2)e^{-\lambda\tau} = 0. \quad (15)$$

Based on the discussion made in section 4.2, we have the linearized dynamic system represented by eq.(14) stable when there is no delay $(\tau_1 = 0, \tau_2 = 0)$. Now assuming $\lambda = i\omega$ with $\omega > 0$ to be the solution of eq.(15) we have:

$$-\omega^2 + i\omega(\alpha_4 + \alpha_1) + (\alpha_1\alpha_4 - \alpha_3\alpha_2)e^{-i\omega\tau} = 0,$$

that leads to

$$\begin{cases} \omega(\alpha_4 + \alpha_1) = (\alpha_1\alpha_4 - \alpha_3\alpha_2)sin(\omega\tau) \\ \omega^2 = (\alpha_1\alpha_4 - \alpha_3\alpha_2)\cos(\omega\tau) \end{cases}. \quad (16)$$

Squaring and adding we obtain:

$$\omega^4 + (\alpha_4 + \alpha_1)^2 \omega^2 - (\alpha_1\alpha_4 - \alpha_2\alpha_3)^2 = 0,$$

from which it is obtained that:

$$\omega_o^{\ 2} = \frac{-(\alpha_4 + \alpha_1)^2 + \sqrt{(\alpha_4 + \alpha_1)^4 + 4(\alpha_1\alpha_4 - \alpha_3\alpha_2)^2}}{2} > 0,$$

and

$$\omega_o = \sqrt{\frac{-(\alpha_4 + \alpha_1)^2 + \sqrt{(\alpha_4 + \alpha_1)^4 + 4(\alpha_1\alpha_4 - \alpha_3\alpha_2)^2}}{2}}.$$

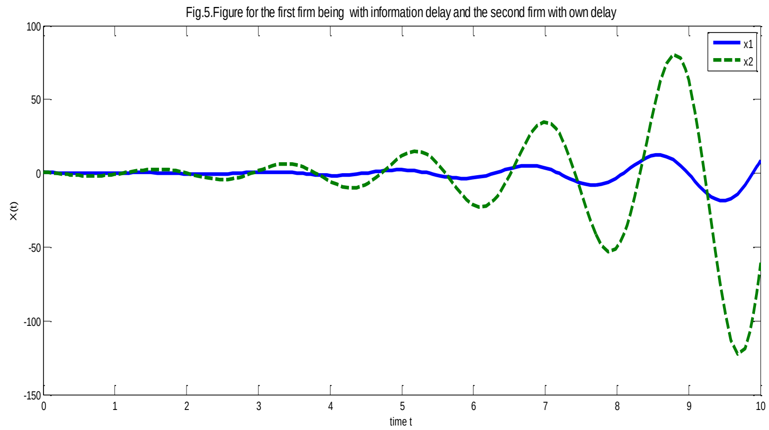
To find τ solve for $\tan(\tau\omega)$ from eq.(16) and obtain:

$$\tau^n = \frac{1}{\omega_o} tan^{-1}\left(\frac{(\alpha_1 + \alpha_4)}{\omega_o}\right) + \frac{n\pi}{\omega_o}, n = 0,1,2,\ldots$$

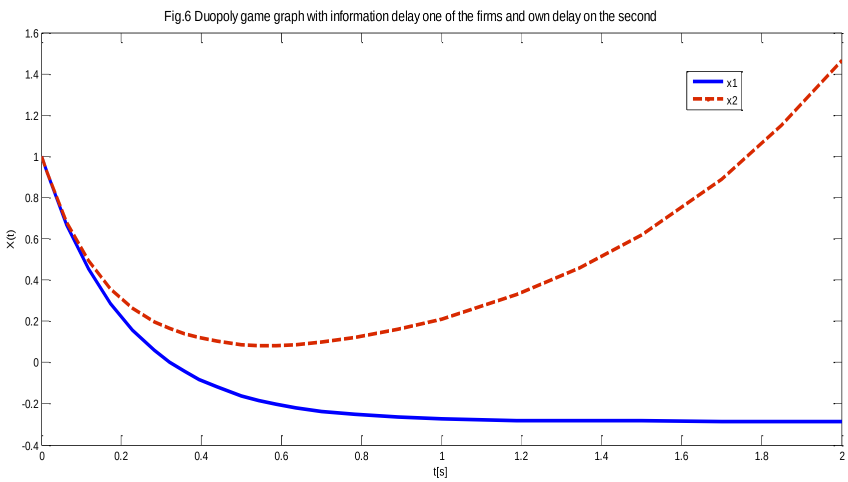
Since the real part of all the roots of the characteristic equation eq.(15) are zero, we can say that $\left(\tau^1 = \frac{1}{\omega_o} tan^{-1}\left(\frac{(\alpha_1 + \alpha_4)}{\omega_o}\right)\right)$ is the smallest (among τ^n) cutoff value at which stability of the equilibrium point is lost and can never be regained in the fiture time t. The numerical experiments shown in fig.5 below is in agreement with the result of this section.

Numerical Simulation III

a). Let $x^* = \left(\frac{1}{3}, \frac{1}{3}\right)$, $b = 1$, $x_1^* = \frac{2b_1 - b_2}{3} > 0$, $x_2^* = \frac{2b_2 - b_1}{3} > 0$, Then $b_1 = b_2 = 2.$ The simulation result for the case where one the firms is with information delay and the second with own (e.g. production) delay is shown in fig.5 below for the same delay constants $(\tau_1 = \tau_2 = 0.5)$.



b). Simulation result for the case where one the firms is with information delay and the second with own (e.g. production) delay is shown in fig.6 below for different delay constants $(\tau_1 = 5, \tau_2 = 10)$.



In both figures 5 and 6,it is clear that the system loses its stability with in few second from the start.

CONCLUSION

In this article, the stability of the Nash equilibrium point of nonlinear Economic model called Duopoly model is analysed. The nonlinear dynamics is described by delay differential equations. The result shows that, global stability of the equilibrium points can be secured in the presence of delay, like it is the case when both the competing firms have information delays. It is also shown that, there are cases when stability is lost after some cutoff point. The results of this article can be used to justify strategies to enhance the competitiveness of production in different enterprises, design best strategies of obtaining optimal profit, keeping the production enterprises from instability, justify problems in the real market and design solution related to production. Future research can be conducted by considering cases such as when one of the competitors has both information and production time delay or one of them has production (own delay) and the other has information delay.

REFERENCES

Agiza, H.N. & Elsadany, A.A. (2003). Nonlinear dynamics in the court duopoly game with heterogeneous players. *Applied mathematics and computation*, 149(3), 843-860.

Almoderesi, E. & Bozorg, M. (2009). Stability Crossing Surfaces for linear time delay systems with three delays. *International journal of Control*, 82, 2304-2310.

Bischi, G.I. & Namzada, A. (2000). Global Analysis of a Dynamic Duopoly Game with Bounded Rationality, in *Advanced in Dynamic Games and Application*, vol.5, Chapter 20, 361-385, Birkhauser, Boston, Masssatusettu, USA.

Elsadany, A.A. (2010).Dynamics of a delayed duopoly game with bounded rationality. *Mathematical and computer modeling*, 52(9-10), 1479-1489.

Kopel, M. (1996). Simple and complex adjustment dynamics in court Duopoly model. *Chaos, Solitons and Fractal*, 7(12), 2013-2048.

Peters, E. (1994). *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics*, John Wiley and Sons.

Smith, H. (2011). *An Introduction to Delay Differential Equations with Applications to the Life Sciences*. Springer New York Dordrecht Heidelberg, London. DOI 10.1007/978-1-4419-7646-8

Xiaoxin, L. Liqiu, W. & P. Yu. (2007). *Stability of Dynamical Systems*, Monograph Series on Nonlinear Science and Complexity, Elsevier B.V.

Zhang, W.B. (2005). *Differential Equations, Bifurcations, and Chaos in Economics, Series on Advances in Mathematics for Applied Sciences, vol.68*, World Scientific publisher.