ORIGINAL ARTICLE

Refined Generalized Iterative Methods in Solving Fuzzy Linear Systems

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Abstract
In this paper refined generalized numerical algorithms for solving systems of linear equations whose coefficient matrices are M-matrices are extended for solving Fuzzy Linear Systems (FLS)such as Refined Generalized Jacobi (RGJ), and Refined Generalized Gauss-Seidel (RGGS) iteration methods. The embedding approach and splitting strategy of the M-matrix together with the refinement process have been employed in the development of these methods. The presented algorithms are tested and compared with a similar work by solving a numerical example and the results showed that the present methods perform better.

Keywords*: Embedding, Fuzzy Linear Systems, Gauss-Seidel, Generalized, Jacobi, Refined*

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INTRODUCTION

Systems of linear equation play a major role in various areas of science. Many problems at various areas of science can be solved by solving a system of linear equation. Since some of the systems are parametric and measurements are vague or imprecise, are represented by fuzzy numbers. One of the major applications of using fuzzy number arithmetic is treating linear systems whose parameters are all or partially represented by fuzzy numbers (Friedman *et al*., 1998). Development of mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them is important. The system of linear equations, $AX = b$, is called fuzzy system of linear equations(FSLE), in which coefficients $(n \times n)$ matrix A is crisp and b is a column matrix which is a fuzzy number vector. A general model for solving a fuzzy linear system (FLS) whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number was first proposed by Friedman *et al*. (1998). They proposed a general model for solving such fuzzy linear systems by using the embedding approach where they replace the original system $AX = b$ by $(2n) \times (2n)$ representation $SX = Y$ .The fuzzy linear equations have been studied by many authors. In following Friedman *et al*. (1998), Allahviranloo*et al*. (2006) and other authors such as Abbasbandy*et al*. (2006), Asady*et al*. (2005), Dehghan*et al*. (2006), Wang *et al*. (2006), Zheng*et al*.(2006), and Ezzati*et al*. (2010) designed some numerical methods for calculating the solution of fuzzy linear system.

Salkuyeh (2007) developed the generalized Gauss-Seidel method for solving non-fuzzy linear systems by using a stationary first order iterative method and the splitting procedure based on M-matrix and concluded that it is more effective than the conventional Jacobi and Gauss-Seidel methods. Refinement of generalized Gauss-Seidel method have been developed by Genanew Gofe (2016) for non-fuzzy linear systems. Various Jacobi based iterative methods using the refinement strategy and over-relaxation parameter for solving non-fuzzy linear systems have been extended to FLS by Abdullah and Rahman (2013). So, one can see that several efforts are exerted in order to extend strategies of solving non-fuzzy linear systems to FLS in an attempt to get more effective methods in terms of having small number of iteration required to converge to the exact solution and less computation times which are applicable in solving FLS.
The purpose of this paper is to extend the works of Genanew (2016) and Abdullah and Rahman (2013) to solve FLS in order to get more accurate and efficient numerical methods than the existing ones.

Preliminaries and Description of the Method
Preliminaries

Definition 2.1: A matrix A is said to be an M-matrix if it satisfies the following four properties

$(i) a_{ii} > 0$, for $i = 1(1)N$
$(ii) a_{ii} < 0$ for $i \neq j, i, j = 1(1)N$
(iii) A is nonsingular
(iii) $A^{-1} \geq 0$

Definition 2.2: A banded matrix is a square matrix with zeros after "m" elements above and below the main diagonal, where m is less than the size of the matrix (i.e. if the matrix is $N \times N$, then $m < N$). In this case where bandedness mater, "m" is usually significantly less than N.

Definition 2.3: Let X denotes a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function $\mu_A: X \to [0,1]$; which assigns a real number $\mu_A(x)$ at x shows the grade of membership x in $\tilde{A}$.

Definition 2.4: A fuzzy set with the following membership function is named a triangular fuzzy number:

$$\mu_A(x) = \begin{cases} 1 - \dfrac{m-x}{\propto}, & m - \propto \leq x \leq m, \propto > 0 \\ 1 - \dfrac{x-m}{\beta}, & m \leq x \leq m+\beta, \beta > 0 \\ 0, & else \end{cases}$$

Definition 2.5: An arbitrary fuzzy number $\tilde{u}$ in parametric form is represented by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right)$, which satisfy the following requirements:

(i) $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over [0,1]
(ii) $\overline{u}(r)$ is a bounded left-continuous non-increasing function over [0,1]
(iii) $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

Definition 2.6: For arbitrary fuzzy numbers $\tilde{x} = \left(\underline{x}(r), \overline{x}(r)\right), \tilde{y} = \left(\underline{y}(r), \overline{y}(r)\right)$ and real number k we define equality by $\tilde{x} = \tilde{y}$, addition by $\tilde{x} + \tilde{y}$ and multiplication as follows:

$(i)\tilde{x} = \tilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$.
$(ii)\tilde{x} + \tilde{y} = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r)$.

$$(iii)\ k\tilde{x} = \begin{cases} \left(k\underline{x}, k\overline{x}\right), & k \geq 0 \\ \left(k\overline{x}, k\underline{x}\right), & k \leq 0 \end{cases}$$

Definition 2.7: The $n \times n$ linear system of equations

$$\begin{cases} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \cdots + a_{1n}\tilde{x}_n = \tilde{b}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \cdots + a_{2n}\tilde{x}_n = \tilde{b}_2, \\ \quad\vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \cdots + a_{nn}\tilde{x}_n = \tilde{b}_n, \end{cases} \tag{2.1}$$

where the coefficient matrix, $A = [a_{ij}]_{i,j=1}^{n}$ is a crisp $n \times n$ matrix and $\tilde{b}_i$ are fuzzy numbers, is called a fuzzy linear system. The matrix form of the system (2.1) is as follows:

$$AXb \tag{2.2}$$

where $X = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ and $b = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)^T$ are the fuzzy number vectors.

Considering the system of equations given in Eq. (2.1) and using splitting procedures in Salkuyeh (2007), we obtain:

$$A = T_m - E_m - F_m \qquad (2.3)$$

where $T_m = (t_{ij})$ is a banded matrix of band width 2m + 1 defined as:

$$t_{ij} = \begin{cases} a_{ij}, 0 \le |i-j| \le m \\ 0, \quad \text{otherwise} \end{cases}$$

Definition 2.8 A fuzzy vector $X = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ that has been given by $\tilde{x}_j = \left(\underline{x}_j(r), \overline{x}_j(r)\right), 1 \le j \le n, 0 \le r \le 1$ is called a solution of (2.1) if:

$$= \sum_{j=1}^{n} \underline{a_{ij}\,\tilde{x}_j} = \underline{b}_i \text{ and } \overline{\sum_{j=1}^{n} a_{ij}\,\tilde{x}_j} = \sum_{j=1}^{n} \overline{a_{ij}\,\tilde{x}_j} = \overline{b}_i;\ i = 1,2,\dots,n \qquad (2.4)$$

Definition 2.9: (Allahviranloo and Hahemi, 2014) For an arbitrary fuzzy number $\tilde{x}$ in parametric form the embedding $x: \Re^2 \to \Re^2$ is defined as follows:

$$\pi\left(\underline{x}(r), \overline{x}(r)\right) = \left(\overline{x}(r) - \underline{x}(r), \overline{x}(r) + \underline{x}(r)\right) \qquad (2.5)$$

Lemma 2.1: (Allahviranloo and Hahemi, 2014) Let $\tilde{x} = \left(\underline{x}(r), \overline{x}(r)\right), \tilde{y} = \left(\underline{y}(r), \overline{y}(r)\right)$ are arbitrary fuzzy numbers and let k is a real number. Then:

$(i)\ \tilde{x} = \tilde{y}$ if and only if $\pi(\tilde{x}) = \pi(\tilde{y})$

$(ii)\ \pi(\tilde{x} + \tilde{y}) = \pi(\tilde{x}) + \pi(\tilde{y})$

$(iii)\ \pi(k\tilde{x}) = \pi\left(k\left(\underline{x}(r), \overline{x}(r)\right)\right) = \pi\left(|k|\left(\overline{x}(r) - \underline{x}(r)\right), k\left(\overline{x}(r) + \underline{x}(r)\right)\right)$

Proof: See Allahviranloo and Hahemi. (2014).

Using previous lemma 2.1 and Eq. (2.1) Allahviranloo and Hahemi (2014) have obtained the following equations:

$$\sum_{j=1}^{n} |a_{ij}|(\overline{x}(r) - \underline{x}(r)) = \overline{b}_i(r) - \underline{b}_i(r), \qquad i = 1,2,\dots,n \qquad (2.6)$$

$$\sum_{j=1}^{n} a_{ij}\,(\overline{x}(r) + \underline{x}(r)) = \overline{b}_i(r) + \underline{b}_i(r), \qquad i = 1,2,\dots,n \qquad (2.7)$$

The matrix form of equations (2.6) and (2.7) are as follows:

$$\mathrm{BU} = \mathrm{Z}, \qquad \mathrm{AY} = \mathrm{W} \qquad (2.8)$$

where $B = [|a_{ij}|]_{i,j=1}^{n}$ and $A = [a_{ij}]_{i,j=1}^{n}$ and the right hand side columns are the vectors $Z = (\overline{b}_1(r) - \underline{b}_1(r)), \dots, (\overline{b}_n(r) - \underline{b}_n(r),)^T, W = (\overline{b}_1(r) + \underline{b}_1(r)), \dots, (\overline{b}_n(r) + \underline{b}_n(r),)^T$, $U = (\overline{x}_1(r) - \underline{x}_1(r)), \dots, (\overline{x}_n(r) - \underline{x}_n(r))^T$ and $Y = (\overline{x}_1(r) + \underline{x}_1(r)), \dots, (\overline{x}_n(r) + xnr)T$ are solutions of the crisp linear system of Equations (2.8).

Moreover, it is stated that the fuzzy linear system (2.1) has a fuzzy solution if the matrices B^-, $B^- - A^-, B^- + A^-$ are nonnegative. Under these conditions, letting the matrices $U = [u_{ij}]_{i,j=1}^{n}$ and $Y = [y_{ij}]_{i,j=1}^{n}$ and solving equations (2.6) and (2.7) for $\overline{x}_1(r)$ and $\underline{x}_1(r)$ we get:

$$\begin{cases} \overline{x}_i(r) = \frac{1}{2}(y_i + u_i) \\ \underline{x}_i(r) = \frac{1}{2}(y_i - u_i) \end{cases} \tag{2.9}$$

It is shown that the equationAX = b can be converted in to two equations as in Eq. (2.6) and Eq. (2.7) obtain its solutions using Eq. (2.9). However, these two equations can be expressed as a single system equations SX = Y as follows

$$\begin{cases} s_{1,1}\underline{x}_1 + \cdots + s_{1,n}\underline{x}_n + s_{1,n+1}(-\overline{x}_1) + \cdots + s_{1,2n}(-\overline{x}_n) = \underline{y}_1, \\ \vdots \\ s_{n,1}\underline{x}_1 + \cdots + s_{n,n}\underline{x}_n + s_{n,n+1}(-\overline{x}_1) + \cdots + s_{n,2n}(-\underline{x}_1) = \underline{y}_n, \\ s_{n+1,1}\underline{x}_1 + \cdots + s_{n+1,n}\overline{x}_n + s_{n+1,n+1}(-\overline{x}_1) + \cdots + s_{n+1,2n}(-\overline{x}_n) = -\overline{y}_1, \\ \vdots \\ s_{2n,1}\underline{x}_1 + \cdots + s_{2n,n}\underline{x}_n + s_{2n,n+1}(-\overline{x}_1) + \cdots + s_{2n,2n}(-\overline{x}_n) = -\overline{y}_n \end{cases} \tag{2.10}$$

where $a_{ij} \geq 0 \Rightarrow s_{i,j} = s_{i+n,j+n} = a_{ij}$, $s_{i+n,j} = s_{i,j+n} = 0$ and
$a_{ij} \leq 0 \Rightarrow s_{i,j+n} = s_{i+n,j} = -a_{ij}$, $s_{i,j} = s_{i+n,j+n} = 0$.

Theorem 2.1 Suppose that $a_{ij} > 0; 1 \leq i \leq n$. The matrix S in (2.10) is strictly diagonally dominant if and only if the matrix A in (2.1) is strictly diagonally dominant.
Proof: See Dehghan and Hashemi (2006).

Theorem 2.2 Let the matrix A in Eq. (2.1) be strictly diagonally dominant with nonnegative diagonal elements, then both the RGJ and RGGS iterative methods converges to $S^{-1}Y$ for any arbitrary initial value X^0.
Proof: See Salkuyeh (2007)

Description of the Methods

Considering the fuzzy linear system of equations $AX = b$ and using Eq. (2.4) we have$\overline{AX} = \overline{b}$and $\underline{AX} = \underline{b}$.

Applying the splitting procedure in Salkuyeh (2007) on A we get:

$(\overline{T}_m - \overline{E}_m - \overline{F}_m)\overline{X} = \overline{b}$(3.1)

and $(\underline{T}_m - \underline{E}_m - \underline{F}_m)\underline{X} = \underline{b}$ (3.2)

Applying the stationary first order Gauss-Seidel and Jacobi iterative methods and refinement procedure on equations (3.1) and (3.2) we obtain RGGS and RGJ iteration formulas respectively.

$$\overline{X}^{k+1} = \left[(\overline{T}_m - \overline{E}_m)^{-1}\overline{F}_m\right]^2 \overline{X}^k + \left[I + (\overline{T}_m - \overline{E}_m)^{-1}\overline{F}_m\right](\overline{T}_m - \overline{E}_m)^{-1} b \tag{3.3}$$

$$\underline{X}^{k+1} = \left[\left(\underline{T}_m - \underline{E}_m\right)^{-1}\underline{F}_m\right]^2 \underline{X}^k + \left[I + \left(\underline{T}_m - \underline{E}_m\right)^{-1}\underline{F}_m\right]\left(\underline{T}_m - \underline{E}_m\right)^{-1} b \qquad (3.4)$$

$$\overline{X}^{k+1} = \left[\overline{T}_m^{\ -1}\left(\overline{E}_m + \overline{F}_m\right)\right]^2 \overline{X}^k + \left[I + \overline{T}_m^{\ -1}\left(\overline{E}_m + \overline{F}_m\right)\right]\overline{T}_m^{\ -1} b \qquad (3.5)$$

$$\underline{X}^{k+1} = \left[\underline{T}_m^{\ -1}\left(\underline{E}_m + \underline{F}_m\right)\right]^2 \underline{X}^k + \left[I + \underline{T}_m^{\ -1}\left(\underline{E}_m + \underline{F}_m\right)\right]\underline{T}_m^{\ -1} b \qquad (3.6)$$

After replacingAX = bintoSX = Yby using Eq. (2.10), Eq. (3.3) and Eq. (3.4) together and Eq. (3.5) and Eq. (3.6) together can be written in one equation as Eq. (3.7) and Eq. (3.8) respectively.

$$X^{k+1} = [(T_m - E_m)^{-1}F_m]^2 X^k + [I + (T_m - E_m)^{-1}F_m](T_m - E_m)^{-1} Y \qquad (3.7)$$

$$X^{k+1} = \left[T_m^{\ -1}(E_m + F_m)\right]^2 X^k + \left[I + T_m^{\ -1}(E_m + F_m)\right]T_m^{\ -1} Y \qquad (3.8)$$

where $S = T_m - E_m - F_m$.

Numerical Example and Results

An example has been considered to verify our methods yield better results than that of Abdullah and Rahman (2013) or not.

Example: Consider the 3x3 FLS given by Dehghan *et. al*. (2007).

$$6x_1 - x_2 - x_3 = (-18 + 16r, 8 - 10r)$$
$$-x_1 + 2x_2 - x_3 = (-8 + 8r, 6 - 6r)$$
$$-x_1 - x_2 + x_3 = (-3 + 4r, 8 - 7r)$$

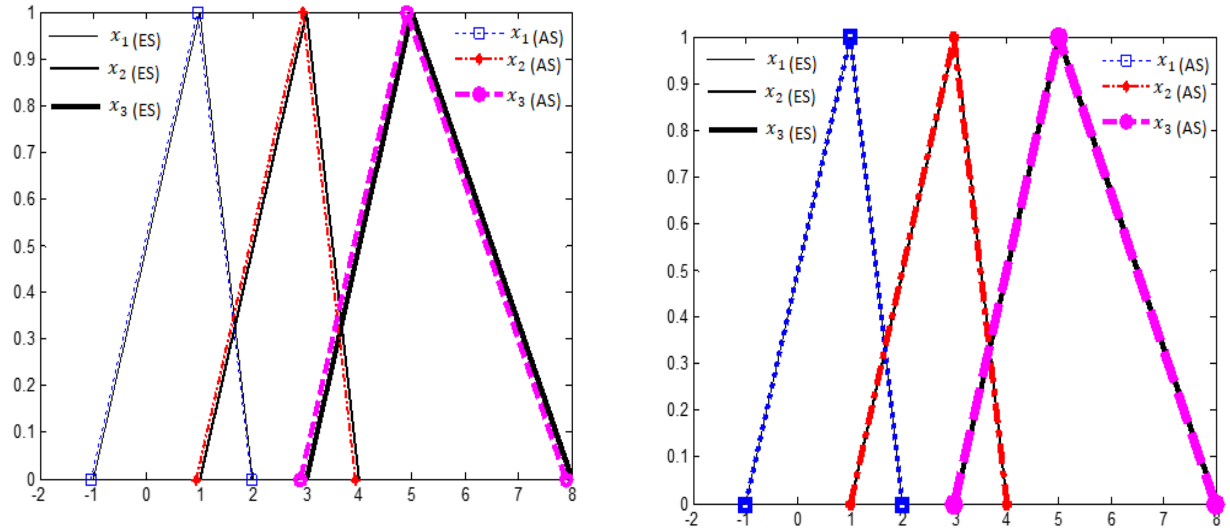
Using equations (2.5) and (2.6) or Eq. (2.9), the exact solution$X = S^{-1}Y$ is $x_1 = (2r - 1, 2 - r)$, $x_2 = (2r + 1, 4 - r)$, and $x_3 = (2r + 3, 8 - r)$.

In order to show which method is better among the three methods and how significantly the Refined Generalized methods improved the refined ones specially the RGGS, the following table (Table 1) is prepared. Moreover, in order to view graphically how the RGGS method converges to the exact solution graphs are sketched as in Fig. 1 for different number of iterations.

For the purpose of comparing the computation times by RJ, RGS, and RGGS methods which required them to converge, a laptop with Intel processor of 1.60 GHz, Installed memory of 4 GB, and 64-bit operating system is used.

Table 1: Comparison of the number of iterations and computation times required by RJ (Abdullah and Rahman, 2013), RGJ, and RGGS methods to converge.

Method	Number of iterations required to converge	Computation times required to converge in seconds
RJ	160	63.823
RGJ	72	10.877
RGGS	21	1.9194



(a) **(b)**

Fig.1: The graphs of the approximated solution (AS) obtained by using RGGS and the exact solution (ES) using the example provided when the number of iterations is: (a) 7 and (b) 15.

It can be observed from Fig. 1 that the solutions obtained by RGGS approaches to the exact solution as the number of iteration increases which ultimately converges after 21 iterations by taking the tolerance level, $\varepsilon = 10^{-3}$.

DISCUSSION AND CONCLUSION

Linear systems have important applications in many branches of science and engineering. In many applications, at least some of the parameters of the system are represented by fuzzy rather than crisp numbers. So it is enormously important to develop numerical procedures that would appropriately treat fuzzy linear systems and solve them.

Salkuyeh (2007) developed the generalized Gauss-Seidel method for solving non-fuzzy linear systems by using a stationary first order iterative method and the splitting procedure based on M-matrix and concluded that it is more effective than the conventional Jacobi and Gauss-Seidel methods.

Genanew Gofe (2016) also showed that the RGGS requires smaller number of iterations to converge than the RGS method for crisp linear systems of equations.
The refinement of the generalized Jacobi and Gauss Seidel methods yield a better result in terms of having a reduced number of iterations to converge to the exact value for FLS which also worked for crisp systems of linear equations. This is due to the fact that the embedding mapping given by Allahviranloo and Hashemi (2014) is one to one and preserves all the properties of the original matrix.

The RGGS method is the best method from the rest methods considered since it requires the smallest number of iterations to converge for all the examples(see Table 1).In particular it yields a much better result than the method proposed by Abdullah and Rahman (2013).So, our result is in agreement with the above mentioned findings.

In terms of computation time required for the methods to converge the RGGS method is an efficient method (see Table 1). Finally, we recommend that this research can be extended to fully fuzzy linear systems.

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