FULL-LENGTH ARTICLE

Coupled Coincidence and Coupled Common Fixed Point Results of Mixed Monotone Mappings in the setting of Partially Ordered Metric Spaces

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ABSTRACT
In this paper, the existence and uniqueness of coupled coincidence and coupled common fixed point of mixed monotone mappings in the setting of partially ordered metric spaces has been proved. Our results extend and generalize several well-known comparable results in the literature. An example is also provided in support of our main result.

Key Words: Coupled coincidence point, coupled common fixed point, mixed monotone mappings, partially ordered metric spaces

INTRODUCTION

Fixed point theory is a powerful tool in modern mathematics. It is also considered to be the key connection between pure and applied mathematics. Its application is not limited to various branches of mathematics but also in many fields such as, Economics, Biology, Chemistry, Physics, Statistics, Computer Science, Engineering etc. This is because in almost all scientific disciplines, most of the problems can be converted into fixed point equations. In other words, the existence of a solution to a theoretical or real-world problem is equivalent to the existence of a fixed point for a suitable map or operator.

The Banach Contraction Principle is the most famous elementary result in the metric fixed point theory and it has fascinated many researchers since 1922. A huge amount of literature contains applications, generalizations and extensions of this principle carried out by several authors in different directions, for example, by weakening the hypotheses, using different setups, considering various types of mappings and generalized form of metric spaces. He developed a theorem called Banach contraction principle which states as follows. Let X be a complete metric space and $T : X \to X$ be a contraction mapping. Then T has a unique fixed point. This principle is one of very useful tools to test the existence and uniqueness of the solution of considerable problems arising in mathematics.

One of the generalizations of Banach contraction principle is in the setting of partially ordered metric spaces given by Ran and Reurings (2004). They generalized Banach contraction principle in partially ordered sets with some applications to matrix equations. Also, Nieto and Lopez (2007) and Agarwal et al. (2008) presented some new results for contractions in partially ordered metric spaces. Bhaskar and Lakshmikantham (2006) initiated the concept of coupled fixed point for non-linear contractions in partially ordered metric spaces. Lakshmikantham and Ciric (2009) established coupled coincidence and coupled common fixed point theorems for two mappings F and g where F has the mixed g-monotone property.

In 2018, Liu et al. established the existence of coupled fixed point for a single mapping satisfying certain contraction condition in a complete partially ordered metric space. Inspired and motivated by the research works of Liu et al. (2018), in this paper we establish new coupled coincidence and coupled common fixed point results for a pair of mixed monotone mappings in the framework of partially ordered complete metric spaces. To the best of our knowledge, there are no similar results in the literature. In this research undertaking, we followed standard procedures.

First, we recall some known definitions and theorems.

Throughout this paper $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^+= [0, +\infty)$, φ denotes all altering distance functions, and Ψ denotes the set of continuous functions such that

$\Psi = \{\psi \in C(\mathbb{R}^+, \mathbb{R}^+) \mid \psi(0) = 0$, and for any $t > 0, \psi(t) > 0\}$.

Definition 1 (Khan et al., 1984). A function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following conditions are satisfied.

(i) φ is continuous and non-decreasing.
(ii) $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2 Let (X, d) be a metric space and $T: X \to X$ be a self-map, then T is said to be a contraction mapping if there exists a constant $k \in [0,1)$ called a contraction factor such that

$$d(Tx, Ty) \leq k\, d(x, y)$$

for all $x, y \in X$.

Definition 3 A set M is said to be partially ordered set if there is a binary relation "$\preccurlyeq$" defined on it such that:

(i) $a \preccurlyeq a$ for all $a \in M$ (Reflexivity);
(ii) If $a \preccurlyeq b$ and $b \preccurlyeq a$ for all $a, b \in M$, then $a = b$ (anti-symmetry);
(iii) If $a \preccurlyeq b$and $\preccurlyeq c$, then $a \preccurlyeq c$ for all $a, b, c \in M$ (Transitivity).

The pair $(M, \preccurlyeq)$ is called partially ordered set.

Note: Two elements $a, b \in M$ are said to be comparable if $a \preccurlyeq b$ or $b \preccurlyeq a$ or both.

Definition 4 Let X be a nonempty set, then $(X, d, \preccurlyeq)$ is said be partially ordered metric space if:

(i) (X, d) is a metric space and
(ii) $(X, \preccurlyeq)$ is a partially ordered set.

Definition 5 (Bhaskar & Lakshmikantham, 2006). Let X be a partially ordered set. A mapping $F: X \times X \to X$ is said to have a mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in X$;

$x_1, x_2 \in X, x_1 \preccurlyeq x_2 \Rightarrow F(x_1, y) \preccurlyeq F(x_2, y)$ and $y_1, y_2 \in X, y_1 \preccurlyeq y_2 \Rightarrow F(x, y_1) \succcurlyeq F(x, y_2)$.

Definition 6 (Bhaskar & Lakshmikantham, 2006). An element $(x, y) \in X \times X$ where X is any nonempty set is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 7 (Lakshmikantham and Ciric, 2009). An element $(x, y) \in X \times X$ is called:

(i) a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$, and (gx, gy) is called coupled point of coincidence.
(ii) a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$.

Definition 8 (Lakshmikantham and Ciric, 2009). Let X be a partially ordered set. A mapping $F : X \times X \to X$ and $g : X \to X$ be two mappings.

(i) We say that F has the g -mixed monotone property if $F(x, y)$ is g monotone non-decreasing in x and non-increasing in y. That is, for any $x_1, x_2 \in X$ and $y_1, y_2 \in X$; $g(x_1) \preccurlyeq g(x_2) \Rightarrow F(x_1, y) \preccurlyeq F(x_2, y)$ and $g(y_1) \preccurlyeq g(y_2) \Rightarrow F(x, y_1) \succcurlyeq F(x, y_2)$.
(ii) Let (X, d) be a metric space, the mappings $F : X \times X \to X$and $g : X \to X$ are called compatible if

$\lim_{n\to\infty} d\big(gF(x_n, y_n), F(gx_n, gy_n)\big) = 0$ and $\lim_{n\to\infty} d\big(gF(y_n, x_n), F(gy_n, gx_n)\big) = 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n$.

Definition 9 (Lakshmikantham and Ciric, 2009). Suppose X is a non-empty set. The mappings $F : X \times X \to X$ and $g: X \to X$ are called commutative if

$$g\big(F(x, y)\big) = F(gx, gy) \text{ and } g(F(y, x)) = F(gy, gx)$$

for all $x, y \in X$.

Definition 10 (Abbas et al., 2010). Suppose X is a non-empty set.The mappings

$F : X \times X \to X$ and $g: X \to X$ are called weakly compatible if

$$g\big(F(x, y)\big) = F(gx, gy) \text{and } g(F(y, x)) = F(gy, gx)$$

whenever $gx = F(x, y)$ and $gy = F(y, x)$.

Theorem 1 (Liu, Mao & Shi, 2018). Assume

(H_1) $\psi \in \Psi$.

(H_2) Let X be a partially ordered metric space and a mapping $F : X \times X \to X$ being a mixed monotone mapping, there exists a constant $k \in (0,1)$ such that:

$$\varphi\big[d\big(F(u, v), F(x, y)\big) + d\big(F(v, u), F(y, x)\big)\big] \le k\varphi\big(d(u, x) + d(v, y)\big) - \psi(k[d(u, x) + d(v, y)])$$

for all $x, y, u, v \in X$ and for each $u \preccurlyeq x$ and $v \succcurlyeq y$. Moreover, φ satisfies

$\varphi(t + s) \leq \varphi(t) + \varphi(s)$, for all $t, s \in \mathfrak{R}^{+}$.

(H_3) There exists $(u_0, v_o) \in X \times X$ such that $u_0 \preccurlyeq F(u_0, v_o)$ and $v_0 \succcurlyeq F(u_0, v_o)$.

(H_4) One of the following conditions holds.

a) F is continuous (or)
b) X has the following properties:
  (i) If a non-decreasing sequence $\{u_n\} \to u$, then $u_n \preccurlyeq u$ for all n;
  (ii) If a non-increasing sequence $\{v_n\} \to v$, then $v_n \succcurlyeq v$ for all n.

Then there exist $u, v \in X$ such that $u = F(u, v)$ and $F(v, u) = v$.

RESULT AND DISCUSSION

Theorem 2 Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ are continuous such that F has the mixed g-monotone property and commutes with g on X such that there exist $x_0, y_0 \in X$ with $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$. The following conditions are satisfied.

(i) $F(X \times X) \subseteq g(X)$.
(ii) There exists $k \in (0,1)$ such that

$$\varphi[d\big(F(x,y), F(u,v)\big) + d\big(F(y,x), F(v,u)\big)] \leq k\varphi((d(gx, gu) + d(gy, gv)) \\ -\psi(k[d(gx, gu)\ d(gy, gv)]) \tag{1}$$

for all $x, y, u, v \in X$ with $gx \succcurlyeq gu$and $gy \preccurlyeq gv, \varphi$ satisfies

$$\varphi(t + s) \leq \varphi(t) + \varphi(s), \text{ for all } t, s \in \mathfrak{R}^{+}.$$

Then F and g have a coupled coincidence point.

Proof. By the hypothesis there exist $x_0 \in X$ and $y_0 \in X$ such that $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, there exist $x_1, y_1 \in X$ such that

$$gx_1 = F(x_0, y_0) \text{ and } gy_1 = F(y_0, x_0).$$

Again from $F(X \times X) \subseteq g(X)$, there exist $x_2, y_2 \in X$ such that

$$gx_2 = F(x_1, y_1) \text{ and } gy_2 = F(y_1, x_1).$$

Continuing this process we can construct sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n)$$

for $n = 0,1,2,\cdots$ and since F has g-monotone property, we have

$$gx_0 \preccurlyeq F(x_0, y_0) = gx_1 \preccurlyeq gx_2 \preccurlyeq \ldots \preccurlyeq F(x_n, y_n) = gx_{n+1} \preccurlyeq \cdots.$$

Similarly $gy_0 \succcurlyeq gy_1 \succcurlyeq gy_2 \ldots \succcurlyeq F(y_n, x_n) = gy_{n+1} \succcurlyeq \cdots$.

If $gx_n = gx_{n+1}$ and $gy = gy_{n+1}$ for some n, then $gx_n = F(x_n, y_n)$ and $gy_n = F(y_n, x_n)$, i.e., (x_n, y_n) is a coupled coincidence point of F and g and this completes the proof.

So, from now on, we assume that $gx_n \neq gx_{n+1}$ and $gy \neq gy_{n+1}$ for $n = 0,1,2,\cdots$. Since $gx_{n-1} \preccurlyeq gx_n$ and $gy_{n-1} \succcurlyeq gy_n$, for $n = 1,2,\cdots$, then from Eq. (1), we have

$$\begin{aligned}
\varphi[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] &= \varphi[d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\
&\quad + d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))] \\
&\leq k\varphi\big(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})\big) \\
&\quad - \psi(k[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})]) \\
&\leq k\varphi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})).
\end{aligned}$$

Since $k \in (0,1)$ and φ is non-decreasing, we have

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \leq d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}). \tag{2}$$

Thus, Eq. (2) holds for each $n \in N$.

Let $\delta_n = d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)$.

It follows that the sequence $\{\delta_n\}$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta.$$

Now, we show that $\delta = 0$.

Suppose on the contrary, that $\delta > 0$.

Since φ is continuous, we have

$$\begin{aligned}
\varphi(\delta) &= lim_{n\to\infty}\ \varphi(\delta_n) \\
&= lim_{n\to\infty}\, \varphi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)) \\
&\leq k\ lim_{n\to\infty}\ \ \varphi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})) \\
&\quad - \lim_{n\to\infty}\ \psi(k[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})]) \\
&\leq k\varphi(\delta) - lim_{n\to\infty}\ \psi(k\delta_{n-1}) \leq k\varphi(\delta) < \varphi(\delta)(\, since\, k \in (0,1)).
\end{aligned}$$

This is a contradiction. Hence $\delta = 0$.

Now, we want to show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Suppose at least $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence, then there exists a positive constant ε such that for any $k > 0$, there exist $n_k > m_k > k$ such that

$$s_k = d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) \geq \varepsilon . \quad (3)$$

Let n_k be the smallest integer satisfying$n_k > m_k > k$ and Eq. (3) holds. Thus

$$d(gx_{n_{k-1}}, gx_{m_k}) + d(gy_{n_{k-1}}, gy_{m_k}) < \varepsilon . \quad (4)$$

From Eq. (3), Eq. (4), and by the triangle inequality, we have

$$\begin{aligned} \varepsilon \leq s_k &= d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) \\ &\leq \mathrm{d}\,(gx_{n_k}, gx_{n_{k-1}}) + \mathrm{d}(gx_{n_{k-1}}, gx_{m_k}) \\ &\quad + d(gy_{n_k}, gy_{n_{k-1}}) + \mathrm{d}(gy_{n_{k-1}}, gy_{m_k}) \\ &= d(gx_{n_{k-1}}, gx_{m_k}) + d(gy_{n_{k-1}}, gy_{m_k}) + \delta_{n_{k-1}} \\ &< \varepsilon + \delta_{n_{k-1}}. \end{aligned} \quad (5)$$

Taking limit as $k \to \infty$ in Eq. (5), we get

$$\varepsilon \leq \lim_{k\to\infty} s_k < \varepsilon + \lim_{k\to\infty} \delta_{n_{k-1}}.$$

Since $\lim_{n\to\infty}\delta_n = 0$, it follows that $\lim_{k\to\infty}\delta_{n_{k-1}} = 0$. Hence $\lim_{k\to\infty}s_k = \varepsilon$.

Again, by the triangle inequality, we have

$$\begin{aligned} s_k &\leq \mathrm{d}(gx_{n_k}, gx_{n_{k+1}}) + \mathrm{d}(gx_{n_{k+1}}, gx_{m_{k+1}}) + \mathrm{d}(gx_{m_{k+1}}, gx_{m_k}) + \mathrm{d}(gy_{n_k}, gy_{n_{k+1}}) \\ &\quad + d(gy_{n_{k+1}}, gy_{m_{k+1}}) + \mathrm{d}(gy_{m_{k+1}}, gy_{m_k}) \\ &= \delta_{n_k} + \delta_{m_k} + \mathrm{d}(gx_{n_{k+1}}, gx_{m_{k+1}}) + \mathrm{d}(gy_{n_{k+1}}, gy_{m_{k+1}}) \end{aligned}$$

Further by the sub-additivity property of φ, we have

$$\varphi\,(s_k) \leq \varphi\,(\delta_{n_k} + \delta_{m_k}) + \varphi\,(d(gx_{n_{k+1}}, gx_{m_{k+1}})) + \varphi\,(d(gy_{n_{k+1}}, gy_{m_{k+1}})).$$

It follows that

$$\begin{aligned} &\varphi\left(d(gx_{n_{k+1}}, gx_{m_{k+1}})\right) + \varphi\left(d(gy_{n_{k+1}}, gy_{m_{k+1}})\right) \\ &\qquad = \varphi\left(d\left(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k})\right)\right) \\ &\qquad + \varphi(d(F(y_{n_k}, x_{n_k}), (F(y_{m_k}, x_{m_k}))) \\ &\qquad \leq k\varphi\,(d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k})) \\ &\qquad -\psi(k[d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k})]). \end{aligned}$$

Thus, we have

$$\begin{aligned} \varphi\,(s_k) &\leq k\varphi\,(s_k) - \psi(ks_k) + \varphi\left(\delta_{n_k} + \delta_{m_k}\right) \\ &\leq \varphi(s_k) - \psi(\mathrm{k}s_k) + \varphi\left(\delta_{n_k} + \delta_{m_k}\right) \qquad (\mathit{since}\ k \in (0,1)) . \end{aligned} \quad (6)$$

Taking limit as $k \to \infty$in Eq. (6), we get

$$\lim_{k\to\infty}\varphi(s_k) \leq \lim_{k\to\infty} \varphi(s_k) - \lim_{k\to\infty}\psi(\,ks_k) + \lim_{k\to\infty}[\varphi\left(\delta_{n_k} + \delta_{m_k}\right)].$$

Since $\delta_n \to 0$, $s_k \to \varepsilon$, and ψ is continuous, we have

$$\varphi(\varepsilon) \leq \varphi(0) + \varphi(\varepsilon) - lim_{k\to\infty}\psi(\,ks_k)$$

$$= \varphi(\varepsilon) - lim_{k\to\infty}\psi(\,ks_k) < \varphi(\varepsilon),$$

which is a contradiction.

Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X.

So, $\lim_{n,m\to\infty} d(gx_n, gx_m) = 0$, $\lim_{n,m\to\infty} d(gy_n, gy_m) = 0$.

Since X is complete, there exist $x, y \in X$ such that

$\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty}F(x_n\,,y_n) = x$, $\lim_{n\to\infty} gy_{n+1} = \lim_{n\to\infty}F(y_n\,,x_n) = y$.

Since F and g are commutative we have

$$g(gx_{n+1}) = g\big(F(x_n\,,y_n)\big) = F(gx_n\,,gy_n). \tag{7}$$

$$g(gy_{n+1}) = g\big(F(y_n\,,x_n)\big) = F(gy_n\,,gx_n). \tag{8}$$

Now, our claim is $gx = F(x,y)$ and $gy = F(y,x)$.

Since F and$g$ are continuous, letting $n \to \infty$ in Eq. (7) and Eq. (8), we get

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty}g\big(F(x_n\,,y_n)\big) = \lim_{n\to\infty}F(gx_n\,,gy_n) = F(x,y),$$

$$gy = \lim_{n\to\infty}g(gy_{n+1}) = \lim_{n\to\infty} g\big(F(y_n\,,x_n)\big) = \lim_{n\to\infty}F(gy_n\,,gx_n) = F(y,x).$$

Hence (gx, gy) is a coupled point of coincidence and (x, y) is a coupled coincidence point of F and g.

Theorem 3 Let all the conditions of Theorem 2 be fulfilled and in addition let the following conditions be satisfied

(i) for every (x, y) and (z, t) in $X \times X$ there exists a (u, v) in $X \times X$ such that $\big(g(u),\ g(v)\big)$ is comparable to both $\big(g(x),\ g(y)\big)$ and $\big(g(z),\ g(t)\big)$.

(ii) F and g are weakly compatible.

Then F and g have a unique coupled common fixed point, that is, there exists a unique

$(x, y) \in X \times X$ such that $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Proof. First we show the uniqueness of coupled point of coincidence of F and g. From Theorem 2 the set of coupled coincidence points of F and g is non-empty. Let (x, y) and (z, t) be coupled coincidence points of F and g that is

$$F(x,y) = gx, F(z,t) = gz, F(y,x) = gy, F(t,z) = gt.$$

Claim: $gx = gz$ and $gy = gt$.

By assumption there exists $(u,v) \in X \times X$ such that $(F(u,v), F(v,u))$ is comparable to

Both $(F(x,y), F(y,x))$ and $(F(z,t), F(t,z))$.

Without loss of generality, we can assume that

$$(F(x,y), F(y,x)) \preccurlyeq (F(u,v), F(v,u)) \text{ and } (F(z,t), F(t,z)) \preccurlyeq (F(u,v), F(v,u)).$$

Put $u_0 = u$ and $v_0 = v$ and by hypothesis there exists $(u_1, v_1) \in X \times X$ suchthat

$$gu_1 = F(u_0, v_0), gv_1 = F(v_0, u_0).$$

For $n \geq 1$, continuing the process we construct sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_{n+1} = F(u_n, v_n) \text{ and } gv_{n+1} = F(v_n, u_n)$$

for all n.

Further set $x_0 = x$, $y_0 = y$, $z_0 = z$ and $t_0 = t$, then on the same way we define sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ and $\{gt_n\}$. Since $(gx, gy) = (F(x,y), F(y,x)) = (gx_1, gy_1)$ and $(F(u,v), F(v,u)) = (gu_1, gv_1)$ are comparable, we have

$$(gx, gy) \preccurlyeq (gu, gv).$$

By induction $(gx_n, gy_n) \preccurlyeq (gu_n, gv_n)$ for all n. Then

$$\begin{aligned} \varphi(d(gx, gu_{n+1}) + d(gy, gv_{n+1})) \\ &= \varphi(d(F(x,y), F(u_n, v_n)) + d(F(y,x), F(v_n, u_n))) \\ &\leq k\varphi(d(gx, gu_n) + d(gy, gv_n)) \\ &\quad -\psi(k[d(gx, gu_n) + d(gy, gv_n)]) \\ &\leq k\varphi\big(d(gx, gu_n) + d(gy, gv_n)\big). \end{aligned} \tag{9}$$

Since φ is non-decreasing and $k \in (0,1)$ we see that

$$d(gx, gu_{n+1}) + d(gy, gv_{n+1}) \leq d(gx, gu_n) + d(gy, gv_n),$$

which implies $\{d(gx, gu_n) + d(gy, gv_n)\}$ is a non-increasing sequence. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} [d(gx, gu_n) + d(gy, gv_n)] = r$.

Taking the limit in Eq. (9) as $n \to \infty$, we get

$$\varphi(r) \leq \varphi(r) - \psi(kr).$$

It follows that $\psi(r) \leq 0$. From the property of ψ we have

$$\psi(r) = 0 \text{ and } r = 0.$$

Therefore, $\lim_{n\to\infty} [d(gx, gu_n) + d(gy, gv_n)] = 0$, which in turn implies that

$$\lim_{n\to\infty} d(gx, gu_n) = 0 \text{ and } \lim_{n\to\infty} d(gy, gv_n) = 0. \tag{10}$$

Similarly, we can prove that

$$\lim_{n\to\infty} d(gz, gu_n) = 0 \text{ and } \lim_{n\to\infty} d(gt, gv_n) = 0. \tag{11}$$

From Eq. (10), Eq. (11) and by the uniqueness of the limit, it follows that $gx = gz$ and $gy = gt$. Hence (gx, gy) is a unique coupled point of coincidence.

Since $gx = F(x, y)$ and $gy = F(y, x)$, by the weakly compatible of F and g, we have

$$g(gx) = g\big(F(x, y)\big) = F(gx, gy) \text{ and } g(gy) = g(F(y, x)) = F(gy, gx).$$

Denoting $gx = a$ and $gy = b$, we get

$$g(a) = F(a, b) \text{ and } g(b) = F(b, a). \tag{12}$$

Thus, (a, b) is a coupled coincidence point of $F$and g.

Then with $z = a$ and $t = b$, it follows that $ga = gx$ and $gb = gy$. That is

$$g(a) = a \text{ and } g(b) = b. \tag{13}$$

From Eq. (12) and Eq. (13), we have

$$a = g(a) = F(a, b).$$

$$b = g(b) = F(b, a).$$

Therefore, (a, b) is a coupled common fixed point of F and g.

To prove the uniqueness of the point (a, b), assume that (c, d) is another coupled common fixed point of F and g, that is,

$$c = gc = F(c, d), d = gd = F(d, c).$$

Since (c, d) is a coupled coincidence point of F and g, we have

$$gc = gx = a \text{ and } gd = gy = b.$$

So, $c = gc = ga = a$ and $d = gd = gb = b$.

Hence the coupled common fixed point is unique.

Theorem 4 Let (X, d) is a partially ordered complete metric space and $F : X \times X \to X$ and $g : X \to X$ are maps where F has the mixed g-monotone property and for $k \in (0,1)$ satisfying

$$\varphi\big[d\big(F(x, y), F(u, v)\big) + d\big(F(y, x), F(v, u)\big)\big] \leq k\varphi\big(d(gx, gu) + d(gy, gv)\big) - \psi(k[d(gx, gu) + d(gy, gv)]) \tag{14}$$

for all $x, y, u, v \in X$ and $gx \succcurlyeq gu$ and $gy \preccurlyeq gv$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose X has the following properties

(a) If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all n.

(b) If a non-increasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all n.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$, then F and g have a coupled coincidence pint.

Proof. In Theorem 2, we have proved that $\{gx_n\}$and $\{gy_n\}$ are Cauchy sequences in X and since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \text{ and } \lim_{n\to\infty} gy_n = y.$$

From the continuity of g, we have

$$\lim_{n\to\infty} g(gx_n) = gx \text{ and } \lim_{n\to\infty} g(gy_n) = gy.$$

Since F and g commute to each other, we have

$$g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n \quad \text{and}$$

$$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

Again $\{ gx_n\}$ is a non-decreasing and $gx_n \to x$, and $\{gy_n\}$ is a non-increasing and

$gy_n \to y$. So we have $gx_n \preccurlyeq x$and $gy_n \succcurlyeq y$. Then by the triangle inequality, we have

$$\begin{aligned}
\varphi(d(gx, F(x,y))) &\le \varphi(d(gx, g(gx_{n+1})) + d(g(gx_{n+1}), F(x,y))) \\
&= \varphi(d(gx, g(gx_{n+1})) + d(F(gx_n, gy_n), F(x,y))) \\
&\le \varphi(d(gx, g(gx_{n+1}))) + \varphi(d(F(gx_n, gy_n), F(x,y))) \\
&\le \varphi(d(gx, g(gx_{n+1}))) + \varphi(d(F(gx_n, gy_n), F(x,y)) \\
&\quad + d(F(gy_n, gx_n), F(y,x))) \\
&\le \varphi(d(gx, g(gx_{n+1}))) + k\varphi(d(g(gx_n), gx) \\
&\quad + d(g(gy_n), gy)) - \psi(k[d(g(gx_n), gx) \\
&\quad + d(g(gy_n), gy)]).
\end{aligned} \tag{15}$$

Taking $n \to \infty$ in Eq. (15), we get

$$\varphi(d(gx, F(x,y)) \le 0.$$

This implies that $F(x,y) = gx$.

Similarly

$$\begin{aligned}
\varphi(d(gy, F(y,x))) &\le \varphi(d(gy, g(gy_{n+1})) + d(g(gy_{n+1}), F(y,x))) \\
&= \varphi(d(gy, g(gy_{n+1})) + d(F(gy_n, gx_n), F(y,x))) \\
&\le \varphi(d(gy, g(gy_{n+1}))) + \varphi(d(F(gy_n, gx_n), F(y,x))) \\
&\le \varphi(d(gy, g(gy_{n+1}))) + \varphi(d(F(gy_n, gx_n), F(y,x)) \\
&\quad + d(F(gx_n, gy_n), F(x,y))) \\
&\le \varphi(d(gy, g(gy_{n+1}))) + k\varphi(d(g(gy_n), gy) \\
&\qquad + d(g(gx_n), gx) - \psi(k[d(g(gy_n), gy) \\
&\qquad + d(g(gx_n), gx)]).
\end{aligned} \tag{16}$$

Taking $n \to \infty$ in Eq. (16), we get

$$\varphi(d(gy, F(y, x)) \leq 0.$$

This implies that $F(y, x) = gy$.

Therefore, (gx, gy) is a coupled point of coincidence and (x, y) a coupled coincidence point of F and g.

Remark: If we take $g = I$ (the identity map), then Theorem 2 will be reduced to Theorem 1.

Example: Let $X = \mathbb{R}$ be a set endowed with the usual order and usual metric

$$d(x, y) = |x - y|$$

for all $x, y \in X$.

$(X, \preccurlyeq)$ is a partially ordered set and $(X, \preccurlyeq, d)$ is a partially ordered metric space.

Define the mappings $F : X \times X \to X$ by $F(x, y) = \frac{x-2y}{8}$ for all $(x, y) \in X \times X$ and $g : X \to X$ by $g\,(x) = \frac{x}{2}$ for all $x \in X$. Then

(i) F and g are continuous.
(ii) For any $x_1, x_2 \in X$ and for all $x, y \in X, gx_1 \preccurlyeq gx_2 \Rightarrow F(x_1, y) \preccurlyeq F(x_2, y)$ and $gy_1 \preccurlyeq gy_2 \Rightarrow F(x, y_1) \succcurlyeq F(x, y_2)$ which implies F has g-monotone property.
(iii) There exist $x_0 = 0$ and $y_0 = 0$ such that $x_0, y_0 \in X$, $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$, that is, $\mathrm{g}(0) = \frac{0}{2} = 0 \preccurlyeq F(0{,}0) = \frac{0-2(0)}{8} = 0$ and $g(0) = \frac{0}{2} = 0 \succcurlyeq F(0{,}0) = \frac{0-2(0)}{8} = 0$.
(iv) $g\big(F(x, y)\big) = g\left(\frac{x-2y}{8}\right) = \frac{x-2y}{16}$ and $F(gx, gy) = F\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{\frac{x}{2}-\frac{2(y)}{2}}{8} = \frac{x-2y}{16}$,

which shows that $g(F(x, y)) = F(gx, gy)$.

In addition

$$g\big(F(y, x)\big) = g\left(\frac{y-2x}{8}\right) = \frac{y-2x}{16} \text{ and } F(gy, gx) = F\left(\frac{y}{2}, \frac{x}{2}\right) = \frac{\frac{y}{2}-\frac{2(x)}{2}}{8} = \frac{y-2x}{16}.$$

which gives that $g(F(y, x)) = F(gy, gx)$.

Hence F and g are commutative.

(i) Let $\varphi(t) = \frac{5t}{4}$, $\psi(t) = \frac{t}{5}$ and $k = \frac{15}{16}$, then

$$d\big(F(x, y), F(u, v)\big) + d\big(F(y, x), F(v, u)\big)$$

$$= \left|\frac{x-2y}{8} - \left(\frac{u-2v}{8}\right)\right| + \left|\frac{y-2x}{8} - \left(\frac{v-2u}{8}\right)\right|$$

$$= \left|\frac{x-u}{8} + \frac{2v-2y}{8}\right| + \left|\frac{y-v}{8} + \frac{2u-2x}{8}\right|$$
$$\leq \frac{1}{8}|x-u| + \frac{1}{4}|y-v| + \frac{1}{8}|y-v| + \frac{1}{4}|u-x|$$

$$= \frac{3}{8}(|x-u|+|y-v|).$$

and then

$$\varphi\left(d\big(F(x,y),F(u,v)\big)+\ d\big(F(y,x),F(v,u)\big)\right)=\frac{5}{4}\big(\frac{3}{8}(|x-u|+|y-v|)\big)$$

$$=\frac{15}{32}(|x-u|+|y-v|),$$

$$d(gx,gu)\ +\ d(gy,gv)\ =\left|\frac{x}{2}-\frac{u}{2}\right|+\left|\frac{y}{2}-\frac{v}{2}\right|=\frac{1}{2}|x-u|+\frac{1}{2}|y-v|=\frac{1}{2}(|x-u|+|y-v|),$$

and

$$k\varphi(d(gx,gu)+\ d(gy,gv))=\left(\frac{15}{16}\right)\left(\frac{5}{4}\right)\big(\frac{1}{2}(|x-u|+|y-v|)\big)$$

$$=\frac{75}{128}(|x-u|+|y-v|).$$

Again, $\psi(k[d(gx,gu)+\ d(gy,gv)])=\frac{1}{5}\left(\frac{15}{32}(|x-u|+|y-v|)\right)=\frac{3}{32}(|x-u|+|y-v|)$.

$$k\varphi\big(d(gx,gu)+\ d(gy,gv)\big)\ -\ \psi(k[d(gx,gu)\ +\ d(gy,gv)])=\frac{75}{128}(|x-u|+|y-v|)-$$

$$-\frac{3}{32}(|x-u|+|y-v|)$$

$$=\frac{63}{128}(|x-u|+|y-v|).$$

Hence

$$\varphi\left(d\big(F(x,y),F(u,v)\big)+\ d\big(F(y,x),F(v,u)\big)\right)=\frac{15}{32}(|x-u|+|y-v|)$$

$$\leq k\varphi\big(d(gx,gu)+\ d(gy,gv)\big)$$

$$-\psi(k[d(gx,gu)\ +\ d(gy,gv)])$$

$$=\frac{63}{128}(|x-u|+|y-v|).$$

Therefore, all the conditions of the Theorem 2 hold and $F$and g have a unique coupled point of coincidence and a unique coupled common fixed point which are $(g0,\ g0)$ and (0,0) respectively. This is because $g(F(0,0))\ =\ F(g0,\ g0)\ =\ F(0,0)\ =\ 0$.

CONCLUSION

In 2018, Liu et al. established the existence of coupled fixed point for mapping satisfying certain contraction condition in a complete partially ordered metric space. In this paper, we have explored the properties of partially ordered metric spaces and established and proved the existence and uniqueness of coupled coincidence and coupled common fixed point results for a pair of mixed monotone mappings satisfying certain contractive condition in the setting of partially ordered metric spaces.

Also, we provided an example in support of our main result. Our work extended coupled fixed point result of a single map to coupled coincidence point and coupled common fixed point results for a pair of maps. The presented theorems extend and generalize several well-known comparable results in literature.

ACKNOWLEGDEMENT

The authors would like to thank the College of Natural Sciences, Jimma University for funding this research work.

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