FULL-LENGTH ARTICLE

A Single Server Markovian Queue with Single Working Vacation, State Dependent Reneging and Retention of Reneged Customers

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ABSTRACT
In this paper an infinite capacity single server Markovian queuing system with single working vacation, state dependent reneging and retention of reneged customers is examined. Whenever a customer arrives at the system, it activates an impatience timer for their service to begun. If it has not begun before the customer's impatience timer expires, then the customers get impatient and may leave the system without getting service with some probability and may remain in the system with probability by employing certain convincing mechanisms for their service. It is assumed that the impatience timer depends on the server states. The closed form expressions of steady state probabilities when the server is in a regular busy period and in a working vacation period are obtained by using probability generating function approach. Various performance measures such as expected system size, expected sojourn time of a customer served, the proportion of customers served and the average reneging rate due to impatience are derived. Finally, some numerical illustrations generated by MATLAB(R2019a) software were presented in order to show how the various parameters of the model influence the performance measures of the system.

Keywords: State Dependent; Single working vacation; CustomerRetention; Reneging; Steady-state solution

INTRODUCTION

In the current scenario of population explosion and globalization of international commerce and trade, queueing problems with customers' impatience have gained a lot of significance in the decision making process. Customers are the backbone of any business. Thus, the concept of customer retention assumes a tremendous importance for the business management (Kumar and Sharma, 2012). Customers are not willing to wait as long as it is necessary to obtain service. As a result, the customers either decide not to join the queue (i.e., Balk) or depart after joining the queue without getting service (i.e., Renege). The significance of this system emerges in many real life situations such as telecommunication systems, call centers, supermarket, production inventory system and etc (Gross *et al.*, 2017).

Queueing models with impatient customers have been extensively studied in the past by various authors. The notion of customer impatience appeared in queuing theory in the work of Haight(1957). He considered a model of balking for the M/M/1 queue in which there was a greatest queue length at which an arrival would not balk. Subsequently, several authors extended these results in various directions. One can refer to

Robert(1979), Abou-El-Ata and Hariri(1992) and Choudhury and Medhi(2010) for related studies.
Keeping in mind the burning problem of customer impatience, the concept of retention of impatient customers has been introduced in queuing modeling by Kumar and Sharma(2012). It is envisaged that the reneged customers may be convinced to stay in the waiting line for their service by employing certain customer retention strategies. Thus, reneged customers may be retained in the queue for their service with probability q and may not be retained with probability $p = (1 - q)$, that is, he may not be convinced and finally decides to leave. They studied the M/M/1/N queue with reneging and retention of reneged customers. Kumar and Sharma(2013) also studied M/M/c/N with reneging and retention of reneged customers. In all aforementioned papers, the source of impatience has always been taken to be either a long wait already experienced upon arrival at a queue, or a long wait anticipated by a customer upon arrival.

However, Altman and Yechiali(2006) have investigated a comprehensive analysis of M/M/1, M/G/1, and M/M/c queue with server vacations and customer impatience, where customers became impatient only when the servers are on vacation. They obtained various closed-form results. Yue *et al*.(2014) extended the M/M/1 model in Altman and Yechiali(2006) to the M/M/1 queueing model with customers' impatience and a variant of multiple vacation policy. Recently, Yue *et al.*(2016) and Boumahdaf (2016) have considered an M/M/1 queueing system with impatient customers with multiple and single vacations. Both assumed that customers are impatient regardless of the states of the server but the two studies differ in their assumption for the impatient timer and rate of impatience distributions when the server is on vacation and regular busy period. The impatient rates and timers are assumed to be different for the two periods for the first case and the same for the second one. In both cases the steady state probabilities and performance measures were obtained by using PGF.

In all of the studies mentioned above, it is assumed that the server stops service during the vacation. However, there are situations where servers continue to provide service with a lower service rate during vacation. Servi and Finn (2002) were the first to introduce this type of vacation, which is called a working vacation. They considered the M/M/1 queue with multiple working vacations (MWVs) and obtained expressions for some performance measures by using the PGF method. Liu et al.(2007) have studied an M/M/1 queue with multiple working vacations by using the matrix geometric method. Analysis of M/M/1 queue with single working vacation was also done by Tian et al. (2008) by applying the same method. Lin and Ke (2009) have extended the study by Tian *et al*.(2008) to the multi-server system. Tian *et al.*(2011) also extended the study by Servi and Finn(2002) to an M/G/1 queue with multiple working vacations. Yue *et al.* (2012) have investigated impatience of customers in M/M/1 queues with multiple working vacations. They considered a model where the customer impatient occurs only during working vacations and derives the closed-form expressions for various performance measures using PGF. Recently, Majid *et al*.(2019) have analyzed single server queue with impatient customers and working vacations. They also considered a model where the customer impatient occurs only during working vacations. Laxmi *et al*.(2019) have considered M/M/1 queuing with single working vacation and reneging of impatient customers in the queue during working vacation period. The stationary probabilities of the model are obtained using PGF.

To the best of our knowledge, for a queueing systems with a working vacation and impatient customers there is no literature which takes reneging and retention of reneged customers into consideration during a regular busy period. Since we are in the world of competitions and globalization, customers prefer the shortest waiting time queue for their service. As a result, customers' impatience may occur due to long wait already experienced in the queue or a long wait anticipated by a customer upon arrival even when the server is on regular busy period. In this paper, we consider a single server Markovian queue with a single working vacation, reneging and retention of reneging customers, where reneging and retention of reneging customers were considered during working vacation and regular busy periods. Our model extends the work of Laxmi et al. (2019) by taking reneging and retention of reneged customers in both regular busy and working vacation periods into account. It also extends the single vaction model of Yue et al. (2016) with inclusion of working vaction and retention of reneged customers regardless of the states of the server.

The model considered in this paper is motivated by some practical application given by Yue *et al.* (2012). Consider a production inventory system where a single product is produced at a single facility to fulfil customer's orders. The production facility can produce ahead of the demand in make to stock fashion. However, the system manager does not want to keep a higher level of inventory of items because more items in inventory result in the increasing of holding costs. As a result, whenever the last order is completed and no order occurs the manager may decide to shut down some machines in the facility to reduce production speed. In other words, the facility produces items with a slow speed for a (random) period of time, which is called a working vacation time. Upon arrival, an order is either fulfilled from the inventory if any production is available or back-ordered. Customers whose orders are back-ordered become impatient and may decide to cancel their orders if the customers' waiting time exceeds a customer's level of patience.

In the example above, customers whose orders are back-ordered may become impatient not only during the working vacation, but also during a regular busy period.Yue *et al*.(2012)] considered only customers impatience during the working vacation. However, in this paper we considered during working vacation and regular busy periods. In addition, customers whose orders are back-ordered may remain in the system with probability q for their service, if certain retention strategies are used and finally may not be convinced, and decides to leave without getting service with probability $p = (1 - q)$. Such system can be modeled by our model considered in this paper.

MODEL DESCRIPTION

We consider an M/M/1 queuing model subject to single working vacation and state dependent retention of reneged customers. Arrivals occur according to a Poisson distribution with a parameter λ, the server provides service according to an exponential distribution with a service rate μ during regular busy period and with slow service rate θ during working vacation periods. Whenever the system becomes empty, the server goes for a single working vacations and the vacation time follows an exponential distribution with a parameter γ. At a vacation completion instant, if there are customers in the system then the server switches back to its regular service rate. Otherwise, itwill stay idle until a new customer arrives.

Whenever a customer arrives at the system and finds that the server is on working vacation (regular busy period), it activates an impatience timer $T_0(T_1)$ for their service to begun, which is exponentially distributed with parameter $\alpha_0(\alpha_1)$. If it has not begun by then, the customers get impatient and may leave the system without getting service with probability p and may remain in the system for their service with probability $q = (1 - p)$. The average reneging rate is $(n-1)\alpha_0$ and $(n-1)\alpha_1$ for $n \geq 1$, where n is the number of customers in the system. The queue discipline is first come first served and the inter-arrival times, the service times, the impatient times and the vacation times are all taken to be identically and exponentially distributed.

Let $L(t)$ denote the number of customers in the system at time t, and let $J(t)$ denote the state of the system at time t, which is defined as follows:

$$J(t) = \begin{cases} 0, & \textit{if the server is in working vacation period} \\ 1, & \textit{if the server is in regular busy period or idle} \end{cases}$$

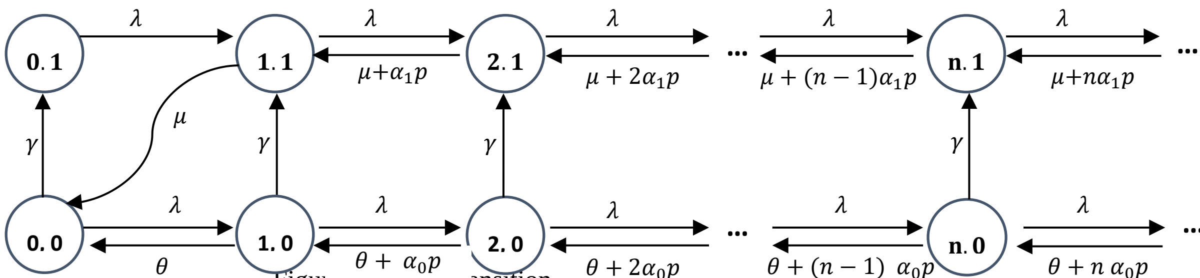
Then, the process $\{L(t), J(t),\ t \geq 0\}$ defines a continuous-time Markov process with state space

$$S = \{(n, j): n = 0,1,2 \dots,\ j = 0,1\}.$$

RESULT AND DISCUSSION

In this section, we study the steady state analysis of our model.

Let $P_{n,j} = \lim_{t\to\infty} P\{L(t) = n, J(t) = j\}, n = 0,1,2, \dots; j = 0,1; (n,j) \in S$ denote the system steady-state probabilities. Then, the state transition diagram for the model is given as follows:



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From the above state transition diagram, the following governing equations of the model are obtained;

$$(\gamma + \lambda)P_{0,0} = \theta P_{1,0} + \mu P_{1,1}$$

$$(\lambda + \gamma + \theta + (n-1)\alpha_0 p)P_{n,0} = \lambda P_{n-1,0} + (\theta + np\alpha_0)P_{n+1,0},\ n \geq 1$$

$$\lambda P_{0,1} = \gamma P_{0,0}$$

$$(\lambda + \mu + (n-1)\alpha_1 p)P_{n,1} = \lambda P_{n-1,1} + \gamma P_{n,0} + (\mu + n\alpha_1 p)P_{n+1,1},\ n \geq 1$$

It is convenient to introduce the probability generating function (PGFs) in order to solve

Equations $(1-4)$.
Define the (partial) probability generating functions $G_0(z)$ and $G_1(z)$ for $0<z<1$,

$$G_0(z)=\sum_{n=0}^{\infty} P_{n,0}z^n \text{ and } G_1(z)=\sum_{n=0}^{\infty} P_{n,1}z^n.$$

With normalizing condition:

$$\sum_{n=0}^{\infty} P_{n,0}+\sum_{n=0}^{\infty} P_{n,1}=1 \quad (5)$$

Denote $G_0'(z)$ and $G_1'(z)$ the respective derivatives of $G_0(z)$ and $G_1(z)$, for $0<z<1$, that is:

$$G_0'(z)=\sum_{n=0}^{\infty} nP_{n,0}z^{n-1} and \; G_1'(z)=\sum_{n=0}^{\infty} nP_{n,1}z^{n-1}$$

Multiplying equation (2) by z^n, and summing all possible values of n, and using (1), we get

$$\alpha_0pz(1-z)G_0'(z)-[(\lambda z-\theta+\alpha_0p)(1-z)+rz]G_0(z)=(\theta-\alpha_0p)(1-z)P_{0,0}-\mu zP_{1,1} \quad (6)$$

In similar manner, multiplying (4) by z^n, and summing all possible values of n and using (3), we get

$$\alpha_1pz(1-z)G_1'(z)-(\lambda z+\alpha_1p-\mu)(1-z)G_1(z)=\frac{r(\mu-\alpha_1p)(1-z)}{\lambda}P_{0,0}-z\gamma G_0(z)+\mu zP_{1,1} \quad (7)$$

Solutions of the differential equations

For $z\neq 0$ and $z\neq 1(6)$ can be written as follows,

$$G_0'(z)-\left(\frac{\lambda}{\alpha_0p}-\frac{\theta}{\alpha_0pz}+\frac{1}{z}+\frac{\gamma}{\alpha_0p(1-z)}\right)G_0(z)=\frac{(\theta-\alpha_0p)}{\alpha_0pz}P_{0,0}-\frac{\mu}{\alpha_0p(1-z)}P_{1,1}$$

To solve first order differential equation (8), an integrating factor can be found as

$$I.F=e^{-\int(\frac{\lambda}{\alpha_0p}-\frac{\theta}{\alpha_0pz}+\frac{1}{z}+\frac{\gamma}{\alpha_0p(1-z)})dz}=e^{-\frac{\lambda}{\alpha_0p}z}z^{\frac{\theta}{\alpha_0p}-1}(1-z)^{\frac{\gamma}{\alpha_0p}}$$

Multiplying both sides of (8) by the integrating factor I.F, we have

$$\frac{d}{dz}\left[e^{-\frac{\lambda}{\alpha_0p}z}z^{\frac{\theta}{\alpha_0p}-1}(1-z)^{\frac{\gamma}{\alpha_0p}}G_0(z)\right]=\left[\frac{(\theta-\alpha_0p)}{\alpha_0p}P_{0,0}\right]e^{-\frac{\lambda}{\alpha_0p}z}z^{\frac{\theta}{\alpha_0p}-2}(1-z)^{\frac{\gamma}{\alpha_0p}}-\left[\frac{\mu}{\alpha_0p}P_{1,1}\right]e^{-\frac{\lambda}{\alpha_0p}z}z^{\frac{\theta}{\alpha_0p}-1}(1-z)^{\frac{\gamma}{\alpha_0p}-1}$$

By Integrating from 0 to z we obtained

$$G_0(z)=\frac{e^{\frac{\lambda}{\alpha_0p}z}z^{1-\frac{\theta}{\alpha_0p}}}{(1-z)^{\frac{\gamma}{\alpha_0p}}}\left[\frac{(\theta-\alpha_0p)}{\alpha_0p}P_{0,0}A_0(z)-\frac{\mu}{\alpha_0p}P_{1,1}A_1(z)\right]$$

Where $A_0(z)=\int_0^z e^{-\frac{\lambda}{\alpha_0p}x}x^{\frac{\theta}{\alpha_0p}-2}(1-x)^{\frac{\gamma}{\alpha_0p}}dx$ and

$$A_1(z) = \int_0^z e^{-\frac{\lambda}{\alpha_0 p}x} x^{\frac{\theta}{\alpha_0 p}-1}(1-x)^{\frac{\gamma}{\alpha_0 p}-1}dx$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{n,0} < \infty$ and $z = 1$ is the root of the denominator of the right hand side of (9), we have that $z = 1$ must be the root of the numerator of the right hand side of (9). So at $z = 1$ we have

$$e^{\frac{\lambda}{\alpha_0 p}z} z^{1-\frac{\theta}{\alpha_0 p}} \left[\frac{(\theta - \alpha_0 p)}{\alpha_0 p} P_{0,0} A_0(z) - \frac{\mu}{\alpha_0 p} P_{1,1} A_1(z)\right] = 0$$

This in turn gives

$$P_{1,1} = \frac{(\theta - \alpha_0 p) A_0(1)}{\mu A_1(1)} P_{0,0} \qquad (10)$$

Substituting (10) in (9), we obtain

$$G_0(z) = e^{\frac{\lambda}{\alpha_0 p}z} z^{1-\frac{\theta}{\alpha_0 p}}(1-z)^{-\frac{\gamma}{\alpha_0 p}} \left[\frac{(\theta-\alpha_0 p)}{\alpha_0 p} A_0(z) - \frac{(\theta-\alpha_0 p)A_0(1)}{\alpha_0 p A_1(1)} A_1(z)\right] P_{0,0}$$

Remark 1: Letting $p = 1, \alpha_0 = \alpha, \theta = \eta$ and $\gamma = \theta$ in (11), we get

$$G_0(z) = e^{\frac{\lambda}{\alpha}z} z^{1-\frac{\eta}{\alpha}}(1-z)^{-\frac{\theta}{\alpha}} \left[\frac{(\eta - \alpha)}{\alpha} F_1(z) - \frac{(\eta - \alpha)F_1(1)}{\alpha F_2(1)} F_2(z)\right] P_{0,0}$$

where $A_0(z) = F_1(z)$ and $A_1(z) = F_2(z)$. This agrees with Laxmi *et al*. (2019) (see Eq.(12), page.4).

In similar manner, for $z \neq 0$ and $z \neq 1$ (7) can be written as follows:

$$G_1'(z) - \left(\frac{\lambda}{\alpha_1 p} + \frac{1}{z} - \frac{\mu}{\alpha_1 p z}\right) G_1(z) = \frac{\gamma}{\lambda} \frac{(\mu-\alpha_1 p)}{\alpha_1 p z} P_{0,0} - \frac{\gamma G_0(z)}{\alpha_1 p(1-z)} + \frac{\mu P_{1,1}}{\alpha_1 p(1-z)}$$

Remark 2: If $p = 1$ and $\alpha_1 = \xi$ in (12), we get

$$G_1'(z) - \left(\frac{\lambda}{\xi} + \frac{1}{z} - \frac{\mu}{\xi z}\right) G_1(z) = \frac{\gamma}{\lambda} \frac{(\mu - \xi)}{\xi z} P_{0,0} - \frac{\gamma G_0(z)}{\xi(1-z)} + \frac{\mu P_{1,1}}{\xi(1-z)}$$

This agrees with Boumahdaf (2016) (see Eq.(14), page.6).

Multiplying both sides of (12) by the integrating factor $I.F = e^{-\frac{\lambda}{\alpha_1 p}z} z^{\frac{\mu}{\alpha_1 p}-1}$, we have

$$\frac{d}{dz}\left[e^{-\frac{\lambda}{\alpha_1 p}z} z^{\frac{\mu}{\alpha_1 p}-1} G_1(z)\right] = \frac{\gamma}{\lambda}\frac{(\mu - \alpha_1 p)}{\alpha_1 p} P_{0,0} e^{-\frac{\lambda}{\alpha_1 p}z} z^{\frac{\mu}{\alpha_1 p}-2} - \frac{\gamma G_0(z)}{\alpha_1 p} e^{-\frac{\lambda}{\alpha_1 p}z}(1-z)^{-1} z^{\frac{\mu}{\alpha_1 p}-1} + \frac{\mu P_{1,1}}{\alpha_1 p} e^{-\frac{\lambda}{\alpha_1 p}z}(1-z)^{-1} z^{\frac{\mu}{\alpha_1 p}-1}$$

By integrating both sides from 0 to z, we have

$$G_1(z) = e^{\frac{\lambda}{\alpha_1 p}z} z^{1-\frac{\mu}{\alpha_1 p}} \frac{\gamma(\mu-\alpha_1 p)}{\lambda\alpha_1 p} P_{0,0} A_2(z) + \frac{\mu P_{1,1}}{\alpha_1 p} A_3(z)$$
$$-\frac{\gamma}{\alpha_1 p}\int_0^z e^{-\frac{\lambda}{\alpha_1 p}x}(1-x)^{-1} x^{\frac{\mu}{\alpha_1 p}-1} G_0(x)dx$$

where $A_2(z) = \int_0^z e^{-\frac{\lambda}{\alpha_1 p}} x^{\frac{\mu}{\alpha_1 p}-2} dx$ and $A_3(z) = \int_0^z e^{-\frac{\lambda}{\alpha_1 p}x}(1-x)^{-1} x^{\frac{\mu}{\alpha_1 p}-1} dx$

Substituting (10) and (11) in (13), we obtain

$$G_1(z) = e^{\frac{\lambda}{\alpha_1 p}z} z^{1-\frac{\mu}{\alpha_1 p}} \left[\frac{\gamma(\mu-\alpha_1 p)}{\lambda\alpha_1 p} A_2(z) + \frac{(\theta-\alpha_0 p)A_0(1)}{\alpha_1 p A_1(1)} A_3(z) - \frac{\gamma(\theta-\alpha_0 p)}{\alpha_1 p \alpha_0 p} A_4(z)\right.$$
$$\left. + \frac{\gamma(\theta-\alpha_0 p)A_0(1)}{\alpha_1 p \alpha_0 p A_1(1)} A_5(z)\right] P_{0,0}$$

where $A_4(z) = \int_0^z e^{\frac{\lambda}{\alpha_0 p}-\frac{\lambda}{\alpha_1 p}x}(1-x)^{-\left(1+\frac{\gamma}{\alpha_0 p}\right)} x^{\frac{\mu}{\alpha_1 p}-\frac{\theta}{\alpha_0 p}} A_0(x)dx$ and

$$A_5(z) = \int_0^z e^{\frac{\lambda}{\alpha_0 p}-\frac{\lambda}{\alpha_1 p}x}(1-x)^{-\left(1+\frac{\gamma}{\alpha_0 p}\right)} x^{\frac{\mu}{\alpha_1 p}-\frac{\theta}{\alpha_0 p}} A_1(x)dx$$

From (11) and (14) both $G_0(z)$ and $G_1(z)$ are expressed in terms of $P_{0,0}$. Thus, once $P_{0,0}$ is calculated, both $G_0(z)$ and $G_1(z)$ are completely determined. We derive $P_{0,0}$ in the next section.

Derivation of $P_{0,0}, P_w$ and P_b

Define $P_w = \sum_{n=0}^{\infty} P_{n,0}$ and $P_b = \sum_{n=0}^{\infty} P_{n,1}$. Then, P_w and P_b represent the probabilities that the server is on working vacation and the server is on regular busy period or idle respectively.

Substituting $z = 1$ in (6) , we have

$$G_0(1) = P_w = \frac{\mu P_{1,1}}{\gamma}$$

Using (10) we get,

$$P_w = \frac{A_0(1)(\theta-\alpha_0 p)}{A_1(1)\gamma} P_{0,0} \qquad (15)$$

Again, by Substituting $z = 1$ in (14)

$$P_b = \frac{e^{\frac{\lambda}{\alpha_1 p}}[A_1 A_2 \gamma\alpha_0 p(\mu-\alpha_1 p)+(\theta-\alpha_0 p)(A_0 A_3 \lambda\alpha_0 p - A_1 A_4 \lambda\gamma + A_0 A_5 \lambda\gamma)]P_{0,0}}{A_1 \lambda\alpha_1 \alpha_0 p^2}$$

where $A_1 = A_1(1)$, $A_2 = A_2(1)$, and so on.

Now, Putting (15) and (16) in (5)$(i.e, \sum_{n=0}^{\infty} P_{n,0} + \sum_{n=0}^{\infty} P_{n,1} = P_w + P_b = 1)$ We obtain,

$$P_{0,0} = \left[\frac{(\theta-\alpha_0 p)A_0(1)}{\gamma A_1(1)} + \frac{e^{\frac{\lambda}{\alpha_1 p}}[A_1A_2\gamma\alpha_0 p(\mu-\alpha_1 p)+(\theta-\alpha_0 p)(A_0A_3\lambda\alpha_0 p-A_1A_4\lambda\gamma+A_0A_5\lambda\gamma)]}{A_1\lambda\alpha_1\alpha_0 p^2}\right]^{-1}$$

Next, the probabilities $P_{n,0}$ and $P_{n,1}$ for $n \geq 1$ can be evaluated in terms of $P_{0,0}$ as below. We obtained expressions by employing the continued fraction, the properties of incomplete gamma function and well-known identities of confluent hyper-geometric function.

Theorem 1: *The probabilities* $P_{n,0}$ *for* $n \geq 1$ *can be expressed in terms of* $P_{0,0}$ *as*

$$P_{n,0} = \left(\frac{\lambda}{\alpha_0 p}\right)^n \frac{P_{0,0}}{\left(\frac{\theta+\gamma}{\alpha_0 p}\right)_n} \frac{{}_1F_1\left(\frac{\theta}{\alpha_0 p}+n; \frac{\theta+\gamma}{\alpha_0 p}+n; -\frac{\lambda}{\alpha_0 p}\right)}{{}_1F_1\left(\frac{\theta}{\alpha_0 p}; \frac{\theta+\gamma}{\alpha_0 p}; -\frac{\lambda}{\alpha_0 p}\right)}$$

where $P_{0,0}$ is given in (17) and $\left(\frac{\theta+\gamma}{\alpha_0 p}\right)_n$ is Pochhammer symbols.

Proof: We can write (2) as follows by taking $\alpha_0 p P_{n,0}$ to divide through the equation and rearranging

$$\frac{P_{n,0}}{P_{n-1,0}} = \frac{\lambda}{\alpha_0 p} \frac{1}{\frac{(\gamma+\theta)}{\alpha_0 p}+(n-1)+\frac{\lambda}{\alpha_0 p}-\left(\frac{\theta}{\alpha_0 p}+n\right)\frac{P_{n+1,0}}{P_{n,0}}} \tag{18}$$

Replacing n by $n+1$ in (18), we get

$$\frac{P_{n+1,0}}{P_{n,0}} = \frac{\lambda}{\alpha_0 p} \frac{1}{\frac{(\gamma+\theta)}{\alpha_0 p}+n+\frac{\lambda}{\alpha_0 p}-\left(\frac{\theta}{\alpha_0 p}+n+1\right)\frac{P_{n+2,0}}{P_{n+1,0}}} \tag{19}$$

Using (19) in (18) and continuing the process, for replacing n by $n+2, n+3, n+4 \ldots$ in (18), we obtained the following continued fractions

$$\frac{P_{n,0}}{P_{n-1,0}} = \frac{\lambda}{\alpha_0 p} \cfrac{1}{\frac{(\gamma+\theta-\alpha_0 p)}{\alpha_0 p}+n-\left(-\frac{\lambda}{\alpha_0 p}\right)+\cfrac{\left(\frac{\theta-\alpha_0 p}{\alpha_0 p}+n+1\right)\left(-\frac{\lambda}{\alpha_0 p}\right)}{\frac{(\gamma+\theta-\alpha_0 p)}{\alpha_0 p}+n+1-\left(-\frac{\lambda}{\alpha_0 p}\right)+\cfrac{\left(\frac{\theta-\alpha_0 p}{\alpha_0 p}+n+2\right)\left(-\frac{\lambda}{\alpha_0 p}\right)}{\frac{(\gamma+\theta-\alpha_0 p)}{\alpha_0 p}+n+2-\left(-\frac{\lambda}{\alpha_0 p}\right)+\cdots}}}$$

By means of the properties of confluent hypergeometric function given by Lorentzen and Waadeland (2008), equation (20) will take the following form,

$$\frac{P_{n,0}}{P_{n-1,0}} = \frac{\lambda}{\alpha_0 p}\frac{1}{\left(\frac{\theta+\gamma}{\alpha_0 p}+n-1\right)}\frac{{}_1F_1\left(\frac{\theta}{\alpha_0 p}+n;\frac{\theta+\gamma}{\alpha_0 p}+\ n;-\frac{\lambda}{\alpha_0 p}\right)}{{}_1F_1\left(\frac{\theta}{\alpha_0 p}+n-1;\frac{\theta+\gamma}{\alpha_0 p}+\ n-1;-\frac{\lambda}{\alpha_0 p}\right)}$$

Putting $n = 1,2,3,4,\ldots$ in (21), yields

$$P_{n,0} = \left(\frac{\lambda}{\alpha_0 p}\right)^n \frac{P_{0,0}}{\left(\frac{\theta+\gamma}{\alpha_0 p}\right)_n}\frac{{}_1F_1\left(\frac{\theta}{\alpha_0 p}+n;\frac{\theta+\gamma}{\alpha_0 p}+n;-\frac{\lambda}{\alpha_0 p}\right)}{{}_1F_1\left(\frac{\theta}{\alpha_0 p};\frac{\theta+\gamma}{\alpha_0 p};-\frac{\lambda}{\alpha_0 p}\right)}$$

Theorem 2: *If* $\xi > 1$ *and* $G_1(z) = \sum_{n=0}^{\infty} P_{n,1}z^n$ *, then for* $0 < z < 1$ *the probabilities* $P_{n,1}$ *can be expressed in terms of* $P_{0,0}$ *and* $P_{k,0}$ *as*

$$P_{n,1} = \frac{A\beta^n}{(\xi-1)(\xi)_n}P_{0,0} + B\sum_{k=0}^{n-1}\frac{\beta^k}{(\xi+n-k)_{k+1}}P_{0,0} - \frac{\gamma}{\alpha_1 p}\sum_{k=0}^{n-1}\phi(k)P_{k,0}$$

where $\phi(k) = \sum_{m=0}^{n-k-1}\frac{\beta^m}{(\xi+n-m-1)_{m+1}}$.

Proof: Substituting (10) in (13)

$$G_1(z) = e^{\beta z}z^{1-\xi}\left[A\ \ P_{0,0}A_2(z) + B\ \ P_{0,0}A_3(z) - \frac{\gamma}{\alpha_1 p}A_4(z)\right]$$

where $A_2(z) = \int_0^z e^{-\beta x}x^{\xi-2}dx, A_3(z) = \int_0^z e^{-\beta x}(1-x)^{-1}x^{\xi-1}dx,$
$A_4(z) = \int_0^z e^{-\beta x}(1-x)^{-1}x^{\xi-1}G_0(x)dx,\ A = \frac{\gamma(\mu-\alpha_1 p)}{\lambda\alpha_1 p}, B = \frac{(\theta-\alpha_0 p)A_0(1)}{\alpha_1 p A_1(1)}, \beta = \frac{\lambda}{\alpha_1 p}$ and $\xi = \frac{\mu}{\alpha_1 p}$.

To have the series expansion of (23), we expand the integrals $A_2(z), A_3(z)$ and $A_4(z)$ in series as below:

According to Gradshteyn and Ryzhik (2007), the integral $A_2(z)$ can be written in terms of incomplete gamma function $\gamma(a,z)$ as

$$A_2(z) = \int_0^z e^{-\beta x}x^{\xi-2}dx = \beta^{1-\xi}\gamma(\xi-1,\beta z) \quad (24)$$

Expressing the incomplete gamma function in terms of Confluent hypergeometric function and expanding the hypergeometric function in series, we have

$$A_2(z) = \frac{z^{\xi-1}e^{-\beta z}}{\xi-1}\sum_{n=0}^{\infty}\frac{\beta^n}{(\xi)_n}z^n \quad (25)$$

Similarly, proceeding with $A_3(z)$ and $A_4(z)$ we obtain

$$A_3(z) = z^{\xi}e^{-\beta z}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\beta^k z^n}{(\xi+n-k)_{k+1}}, A_4(z) = z^{\xi}e^{-\beta z}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\phi(k)P_{k,0}z^{n-1}$$

where $\phi(k) = \sum_{m=0}^{n-k-1} \frac{\beta^m}{(\xi+n-m-1)_{m+1}}$ and $P_{k,0}$ given in (22).

Using (25) and (26) in (23) we get,

$$G_1(z) = \frac{A}{\xi - 1} P_{0,0} \sum_{n=0}^{\infty} \frac{\beta^n z^n}{(\xi)_n} + BP_{0,0} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{\beta^k z^n}{(\xi + n - k - 1)_{k+1}} - \frac{\gamma}{\alpha_1 p} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \phi(k) P_{k,0} z^n (27)$$

Since $G_1(z) = \sum_{n=0}^{\infty} P_{n.1} z^n$, then comparing the coefficients of z^n in (27), yields

$$P_{n,1} = \frac{A\ \beta^n}{(\xi-1)(\xi)_n} P_{0,0} + B \sum_{k=0}^{n-1} \frac{\beta^k}{(\xi+n-k)_{k+1}} P_{0,0} - \frac{\gamma}{\alpha_1 p} \sum_{k=0}^{n-1} \phi(k) P_{k,0}$$

The Performance Measures

In this subsection, we present some important performance measures of the model.

The Expected system size during working vacation period

It is denoted by $E(L_w)$, given as

$$E(L_w) = \sum_{n=1}^{\infty} nP(J = 0, L = n) = G_0'(1)$$

From (6) we have,

$$E(L_w) = \lim_{z \to 1} \frac{[(\lambda z - \theta + \alpha_0 p)(1-z) + \gamma z] G_0(z) + (\theta - \alpha_0 p)(1-z) P_{0,0} - \mu z P_{1,1}}{\alpha_0 p z (1-z)}$$

Since the right-hand side of (29) is indeterminate of the form zero/zero at $z = 1$ Using L' Hospital's rule, we have

$$E(L_w) = \frac{(\lambda - \theta + \alpha_0 p) P_w + (\theta - \alpha_0 p) P_{0,0}}{\alpha_0 p + \gamma} \quad (30)$$

where P_w and $P_{0,0}$ is given in (15) and (17) respectively.

The Expected system size during regular busy period

It is denoted by $E(L_b)$, given as

$$E(L_b) = \sum_{n=1}^{\infty} nP_{n,1} = G_1'(1)$$

From (7), we have

$$E(L_b) = \lim_{z \to 1} \left[\frac{(\lambda z + \alpha_1 p - \mu)(1-z) G_1(z)}{z(1-z)\alpha_1 p} + \frac{\frac{\gamma}{\lambda}(\mu - \alpha_1 p)(1-z) P_{0,0}}{z(1-z)\alpha_1 p} - \frac{z\gamma G_0(z) + \mu z P_{1,1}}{z(1-z)\alpha_1 p} \right]$$

Using L' Hospital's rule, we obtain

$$E(L_b) = \frac{\lambda \gamma E(L_w) + \gamma(\mu - \alpha_1 p) P_{0,0} + \lambda(\lambda + \alpha_1 p - \mu) P_b}{\lambda \alpha_1 p} \quad (32)$$

where $E(L_w), P_{0,0}$ and P_b are given in (30),(17) and (16) respectively.

Therefore, denote $E(L)$ the expected system size, then

$$E(L) = E(L_w) + E(L_b) \qquad (33)$$

The Sojourn times

Let S be the total sojourn time of a customer in the system, measured from the moment of arrival until departure, that departure either being due to completion of service or as a result of abandonment. Then, by little's law, the expected total sojourn time of a customer in the system $E(S)$ is given as

$$E(S) = \frac{E(L)}{\lambda} \qquad (34)$$

However, a more important measure of performance is S_{served}, defined as the total sojourn time of a customer who completes his service. Denote $S_{j,n} = E(S|X_0 = (j, n + 1))$ the conditional sojourn time of a tagged customer in the system who does not abandon the system, given that the state upon his arrival is (j, n), because the tagged customer is included in the system. We use a first-step analysis method which consists of considering what the Markov chain does at time 1, i.e. after it takes one step from its current position. The total sojourn time when the tagged customer upon his arrival is (1,0), i.e. the server is idle, is given by

$$E(S_{1,0}) = \frac{1}{\mu} \qquad (35)$$

Now, for $n = 1,2 \ldots$ we derive $E(S_{1,n})$ by using the method used by Boumahdaf (2016) by conditioning on whether the next transition is a departure (either because of completion service or an impatient customer) or an arrival, for $n \geq 1$ we obtain,

$$E(S_{1,n}) = E(S|X_0 = (1, n + 1))$$

$$= \frac{\mu + n\alpha_1 p}{a_n}\left(E(S|X_0 = (1, n + 1), X_1 = (1, n)\right) + \frac{\lambda}{a_n} E(S|X_0 = (1, n + 1), X_1 = (1, n + 2))$$

$$= \frac{\mu + n\alpha_1 p}{a_n}\left(E(S|X_0 = (1, n)) + \frac{1}{a_n}\right) + \frac{\lambda}{a_n}\left(E(S|X_0 = (1, n + 1)) + \frac{1}{a_n}\right)$$

$$= \frac{\mu + n\alpha_1 p}{a_n}(E(S|X_0 = (1, n))) + \frac{\lambda}{a_n} E(S_{1,n}) + \frac{1}{a_n}$$

where $a_n = \lambda + \mu + n\alpha_1 p$.

Now, by considering the probability $\frac{n-1}{n}$ that, when there is an abandonment among n waiting customers, our customer will not be the one to leave. We have

$$\frac{\mu+n\alpha_1 p}{a_n}\left(E(S|X_0=(1,n)\right)=\frac{\mu}{a_n}E\left(S_{1,n-1}\right)+\frac{(n-1)\alpha_1 p}{a_n}E\left(S_{1,n-1}\right).$$

This yields

$$E\left(S_{1,n}\right)=\frac{\mu+(n-1)\alpha_1 p}{a_n}E\left(S_{1,n-1}\right)+\frac{\lambda}{a_n}E\left(S_{1,n}\right)+\frac{1}{a_n}.$$

By rearranging, we obtain

$$E\left(S_{1,n}\right)=\frac{\mu+(n-1)\alpha_1 p}{\mu+n\alpha_1 p}E\left(S_{1,n-1}\right)+\frac{1}{\mu+n\alpha_1 p} \qquad (36)$$

By using (35) and iterating (36) for $n=1,2,3,\dots$, we obtain

$$E\left(S_{1,n}\right)=\frac{n+1}{\mu+n\alpha_1 p},\ n\geq\ 0 \qquad (37)$$

Similarly, for $n=0,1,2\dots$ we derive $E\left(S_{0,n}\right)$ using the method used by Yue al.(2012) by conditioning on whether the next transition is a departure (either because of completion service or an impatient customer) or an arrival or vacation completion. For $n\geq 1$ we have,

$$E\left(S_{0,n}\right)=E(S|X_0=(0,n+1))$$

$$=\frac{\gamma}{b_n}\left(E(S|X_0=(1,n+1)+\frac{1}{b_n}\right)+\frac{\lambda}{b_n}\left(E(S|X_0=(0,n+1)+\frac{1}{b_n}\right)$$
$$+\frac{\theta+(n-1)\alpha_0 p}{b_n}\left(E(S|X_0=(0,n)+\frac{1}{b_n}\right)$$

Hence, after some calculations we obtain

$$E\left(S_{0,n}\right)=\frac{\gamma+\lambda+\theta+(n-1)\alpha_0 p}{{b_n}^2}+\frac{\gamma}{b_n}E\left(S_{1,n}\right)+\frac{\lambda}{b_n}E\left(S_{0,n}\right)+\frac{\theta+(n-1)\alpha_0 p}{b_n}E\left(S_{0,n-1}\right)$$

where $b_n=\lambda+\gamma+n\alpha_0 p+\theta$, for $n=1,2,\dots$

Substituting (37) in (38), we obtain

$$E\left(S_{0,n}\right)=\phi_n+\psi_n E\left(S_{0,n-1}\right),\ n\ \geq 1 \qquad (39)$$

where $\phi_n=\frac{1}{\gamma+n\alpha_0 p+\theta}\left(\frac{b_{n-1}}{b_n}+\frac{\gamma(n+1)}{\mu+n\alpha_1 p}\right)$ and $\psi_n=\frac{\theta+(n-1)\alpha_0 p}{\gamma+n\alpha_0 p+\theta}$.

For $n=0$, we have that,

$$E\left(S_{0,0}\right)=\frac{1}{\gamma+\theta}\left(1+\frac{\gamma}{\mu}\right) \qquad (40)$$

By iterating (39) for $n=1,2,\dots$ we obtain that,

$$E(S_{0,n}) = \phi_n + \sum_{i=2}^{n} \phi_{i-1} \prod_{j=i}^{n} \psi_j + \prod_{j=1}^{n} \psi_j E(S_{0,0})$$

Finally, the expected sojourn time of a customer that is served can be calculated by using the expression:

$$E(S_{served}) = \sum_{n=0}^{\infty} P_{n,1} E(S_{1,n}) + \sum_{n=0}^{\infty} P_{n,0} E(S_{0,n})$$

The Other performance measures

The proportion of customers served, denoted by P_s, the average reneging rate due to impatience, denoted by R_r and the proportion of lost customers denoted P_l. Clearly, the expected number of customers served per unit of time is given by

$$\mu \sum_{n=1}^{\infty} P_{n,1} + \theta \sum_{n=1}^{\infty} P_{n,0}$$

Hence, the proportion of customers served P_s is given by

$$P_s = \frac{\mu \sum_{n=1}^{\infty} P_{n,1} + \theta \sum_{n=1}^{\infty} P_{n,0}}{\lambda} = \frac{\lambda\mu P_b + \theta\lambda P_w - (\mu\gamma + \theta\lambda) P_{0,0}}{\lambda^2} \qquad (43)$$

The average reneging rate R_r during working vacation period and regular busy period is given by

$$R_r = \sum_{n=1}^{\infty} (n-1)\alpha_0 p P_{n,0} + \sum_{n=1}^{\infty} (n-1)\alpha_1 p P_{n,1}$$

This implies that,

$$R_r = \frac{\alpha_0 p\lambda E(L_w) + \alpha_1 p\lambda E(L_b) - \lambda(\alpha_0 p P_w + \alpha_1 p P_b) + (\lambda\alpha_0 p + \alpha_1 p\gamma) P_{0,0}}{\lambda}$$

The proportion of lost customers P_l can be obtained by

$$P_l = \frac{R_r}{\lambda} = \frac{\alpha_0 p\lambda E(L_w) + \alpha_1 p\lambda E(L_b) - \lambda(\alpha_0 p P_w + \alpha_1 p P_b) + (\lambda\alpha_0 p + \alpha_1 p\gamma) P_{0,0}}{\lambda^2}$$

Numerical Analysis

In this subsection, we present some numerical examples to demonstrate how the various parameters of the model influence the performance measures of the system and few of those are presented in the form of tables and graphs.

Table 1 shows the impact of $q = (1-p)$ on the system probabilities. The considered parameters are $\lambda = 2, \mu = 3, \theta = 2.5, \gamma = 0.8, \alpha_0 = 0.9$ and $\alpha_1 = 0.6$. From Table 1 we can see that, the probability $P_{0,0}$, $P_{1,0}$ and $P_{2,0}$ decreases with the increasing of parameter q. Consequently, the probability that the system is on working vacation period P_w is reduced. Moreover, probability P_{idle} and $P_{1,1}$ decrease with the increasing of q whereas $P_{2,1}$ slowly increases with retention probability q, this leads to an increase in the probability that the system is on regular busy period P_b. This follows from the fact that retaining of customers in the system results in increasing the tendency or probabilities that the system stays on a regular busy period. In the other hand, the system continues service with fast rate and never goes for a working vacation.

Table 1: Impact of the retention probability q on system probabilities

q	$P_{0,0}$	$P_{1,0}$	$P_{2,0}$	P_w	P_{idle}	$P_{1,1}$	$P_{2,1}$	P_b
0	0.29184	0.15246	0.065978	0.54503	0.11674	0.14534	0.10313	0.45497
0.1	0.28799	0.14977	0.043173	0.53992	0.11519	0.14398	0.11086	0.46008
0.2	0.28380	0.14689	0.026832	0.53426	0.11352	0.14247	0.11784	0.46574
0.3	0.27924	0.14380	0.015624	0.52796	0.11170	0.14079	0.12403	0.47204
0.4	0.27424	0.14046	0.008355	0.52088	0.10969	0.13890	0.12937	0.47912
0.5	0.26871	0.13684	0.003979	0.51288	0.10749	0.13677	0.13379	0.48712
0.6	0.26258	0.13289	0.001603	0.50377	0.10503	0.13434	0.13723	0.49623
0.7	0.25577	0.12858	0.000497	0.49341	0.10231	0.13158	0.13962	0.50659
0.8	0.24398	0.12177	0.000093	0.47341	0.09759	0.12624	0.14020	0.52659

Table 2 shows the impact of impatience rate during working vacation period α_0 and $q = (1-p)$ on system performance measures. The parameters are taken as $\lambda = 2, \mu = 3, \theta = 2.5, \gamma = 0.5$ and $\alpha_1 = 0.7$. It is observed that for fixed α_0, as expected $E(L_w)$, $E(L_b)$, $E(L)$, $E(S)$ and P_s increase as q increases while R_r and P_l decrease as q increases. Further for fixed q , increasing α_0 result in the decrease of $E(L_w), E(L_b), E(L), E(S)$ and P_s while R_r and P_l increase with the increase of α_0. This shows queue system without reneging $(as\ q \to 1)$ is better than queue system with the retention of reneged customers and queue system with simple reneging $(as\ q = 0)$.

Table 2: Impact of α_0 and q on performance measures of the system

q	α_0	$E(L_w)$	$E(L_b)$	$E(L)$	$E(S)$	P_s	R_r	P_l
0	0.7	0.58889	0.53738	1.1263	0.56314	0.81616	0.36769	0.18384
	0.8	0.57403	0.52972	1.1038	0.55188	0.81101	0.37798	0.18899
	0.9	0.56074	0.52294	1.0837	0.54184	0.80626	0.38749	0.19374
0.4	0.7	0.62122	0.69473	1.3160	0.65798	0.85511	0.28977	0.14489
	0.8	0.60865	0.68530	1.2939	0.64697	0.85088	0.29823	0.14912
	0.9	0.59715	0.67677	1.2739	0.63696	0.84690	0.30620	0.15310
0.8	0.7	0.65964	1.20360	1.8632	0.93162	0.91606	0.16788	0.08394
	0.8	0.65304	1.19260	1.8457	0.92283	0.91386	0.17229	0.08614
	0.9	0.64674	1.18230	1.8290	0.91451	0.91171	0.17657	0.08829

Table 3: Impact of α_1 and q on performance measures of the system

q	α_1	$E(L_w)$	$E(L_b)$	$E(L)$	$E(S)$	P_s	R_r	P_l
0	0.6	0.55551	0.55592	1.1114	0.55571	0.81273	0.37453	0.18727
	0.7	0.56074	0.52294	1.0837	0.54184	0.80626	0.38749	0.19374
	0.8	0.56535	0.49565	1.0610	0.53050	0.80055	0.39889	0.19945
0.4	0.6	0.59167	0.71868	1.3104	0.65518	0.85289	0.29422	0.14711
	0.7	0.59715	0.67677	1.2739	0.63696	0.84690	0.30620	0.15310
	0.8	0.60210	0.64218	1.2443	0.62214	0.84149	0.31702	0.15851
0.8	0.6	0.64336	1.27730	1.9207	0.96035	0.91478	0.17044	0.08522
	0.7	0.64674	1.18230	1.8290	0.91451	0.91171	0.17657	0.08829
	0.8	0.65019	1.11090	1.7611	0.88056	0.90857	0.18286	0.09143

Table 3 shows the impact of customers' impatience rate during regular busy α_1 and q on system performance measures. The parameters are taken as $\lambda = 2, \mu = 3, \theta = 2.5, \gamma = 0.5$ and $\alpha_0 = 0.9$. From Table 3 we can see that for a fixed α_1, as intuitively expected $E(L_w)$, $E(L_b)$, $E(L)$, $E(S)$ and P_s increase whereas R_r and P_l decrease as q increases. Moreover for fixed q, increasing α_1 results in the decrease of $E(L_b), E(L), E(S)$ and P_s while $E(L_w)$ increase as α_1 increase. This is due to the fact that reneging of customers during a regular busy period decreases the queue size of this period, as a result the server will go for working vacation period rapidly and thereby starts to provide service slowly. This contributes for $E(L_w)$ to increase. Obviously for a fixed q, R_r and P_l increase as α_1 increases.

Figure 2 depicts the impact of arrival rate λ on the expected system size $E(L)$ and the proportion of customers served P_s for various values of q. The considered parameters are $\mu = 3, \theta = 2.5, \gamma = 1.5, \alpha_1 = 0.2$ and $\alpha_0 = 0.6$. From Figure 2 we observed that for fixed q, as λ increase $E(L)$ increase and P_s decrease. It is consistent with our intuition that the more customers there are in the system, the greater chance that the number of customers renege and leave the system, this leads to a decrease in the number of customers served. Further, it may observed that for a fixed λ, $E(L)$ and P_s increase as the probability of retaining impatient customers q increases.

Figure 3 shows the impact of vacation rate γ on the $E(L)$ and the expected sojourn time of the system $E(S)$, for various α_0. The parameters are taken as $\lambda = 2, \mu = 3, \theta = 2.5, \alpha_1 = 1.3$ and $q = 0.2$. From Figure 3 we see that for fixed α_0, as γ increase $E(L)$ and $E(S)$ decrease. This is due to the fact that larger the vacation rate implies shorter the vacation duration, so that the probability that the customer is served by regular service rate (fast rate) increases. As a result, customers are served and leave the system quickly. This shows that, increasing of vacation rate has a positive effect in the system. Further, for any γ as α_0 increase, $E(L)$ and $E(S)$ decrease.

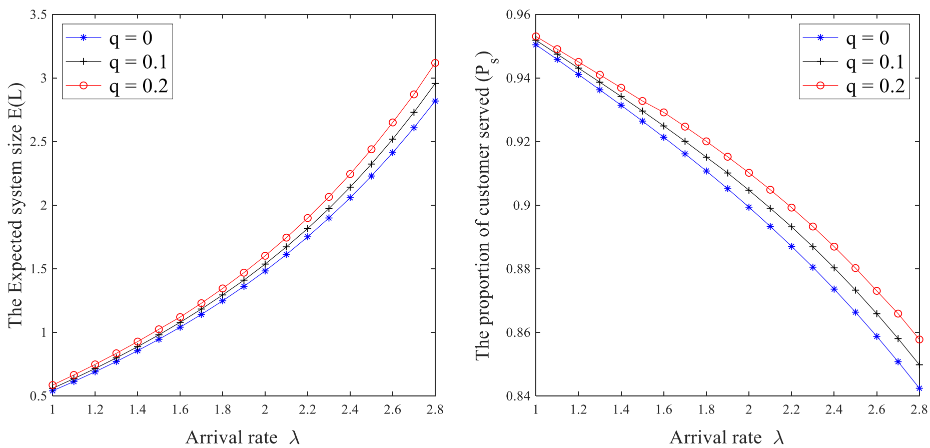

Figure 2: Impact of arrival rate λ on $E(L)$ and P_s

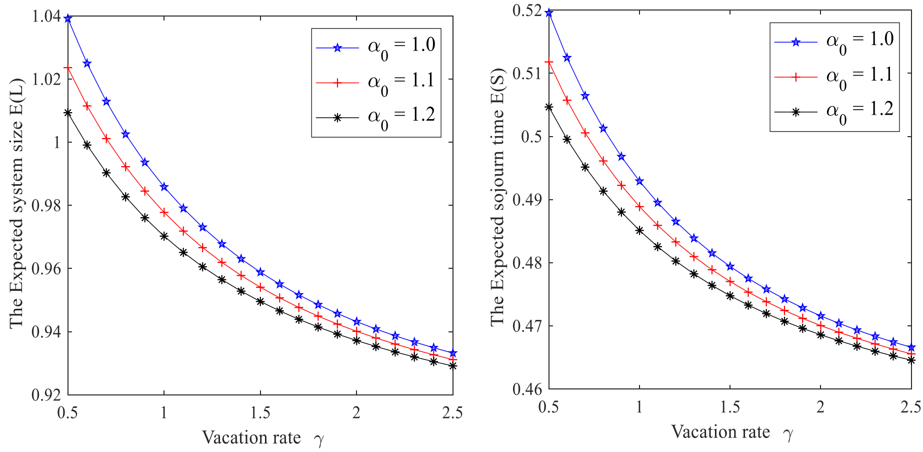

Figure 3: Impact of vacation rate γ on $E(L_b)$ and $E(L_w)$

Figure 4 present the impact of service rate during working vacation θ on the $E(L)$ and P_s for various values of q. The considered parameters are $\lambda = 1, \mu = 2, \gamma = 0.1, \alpha_1 = 0.9$ and $\alpha_0 = 0.2$. From Figure 4 it is clearly observed that, for a fixed θ as q increases $E(L)$ and P_s increase. However for a fixed q as θ increases $E(L)$ decreases and P_s increases. This is from the logical that, higher service rate implies faster service, hence results in higher customers served per unit time and the small queue size.

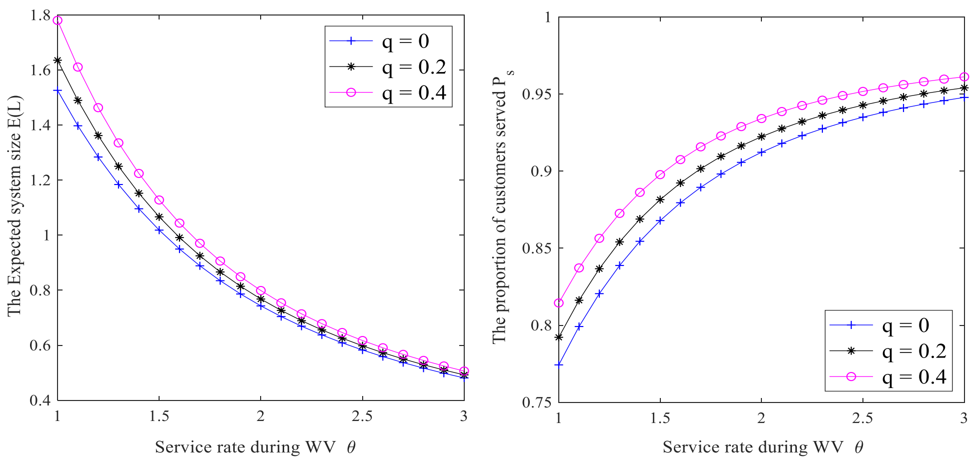

Figure 4: Impact of service rate θ on $E(L)$ and P_s

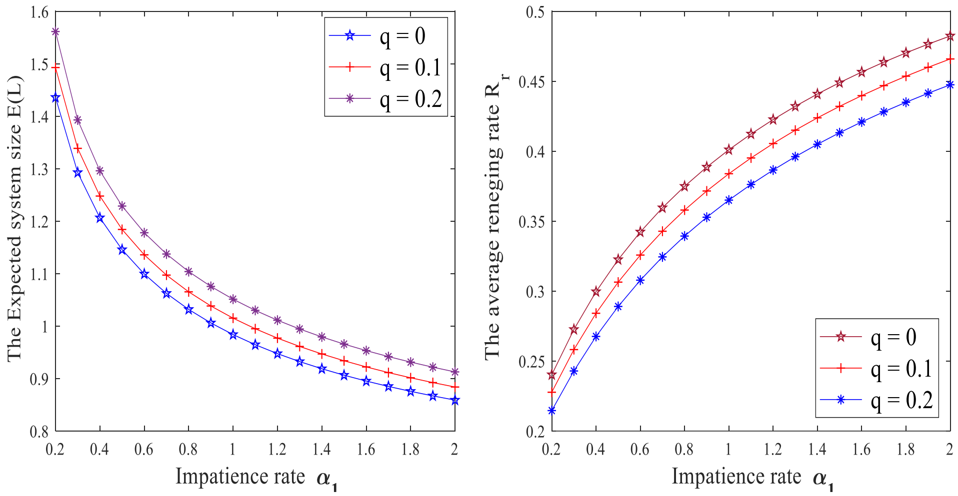

Figure 5: Impact of impatience rate α_1 on $E(L)$ and R_r

Figure 5 show the impact of impatient rate during regular busy period α_1 on $E(L)$ and R_r, for various values of q. The considered parameters are $\lambda = 2, \mu = 3, \theta = 2.5, \gamma = 1$ and $\alpha_0 = 0.9$. From Figure 5 as expected, $E(L)$ decreases and R_r increases as α_1 increases for fixed q. However, it observed that the above effects get reversed when q increases for a fixed α_1.

CONCLUSION

In this paper, we have studied an M/M/1 queue with single working vacation, reneging and retention of reneging customers, where customer reneging timers depend on the states of the server. The steady state probabilities of the system are obtained, using probability generating functions (PGFs). The important performance measures of the system such as the expected system size when the server is on regular busy period and working vacation, the expected sojourn time of a customer served and other performance measures are derived. Numerical results in the form of tables and graphs are presented to display the impact of the model parameters on the system performance measures. Our system can be considered as a generalized version of the existing queueing models given by Yue et al.(2016) and Laxmi et al.(2019).The model considered in this paper can be extended to multi server queueing system with state dependent customers impatience timers and service times.

REFERENCES

Abou-El-Ata, M.O., & Hariri, A. M. A. (1992). The M/M/c/N queue with balking and reneging.*Computers and operations research*, 19(8), 713-716.

Altman, E., & Yechiali, U. (2006). Analysis of customer's impatience in queue with servervacations. *Queueing Systems*, 52, 261-279.

Ancker Jr, C.J., & Gafarian, A.V. (1963). Some queue problems with balking and reneging. *Operations Research*, 11(1), 88-100.

Boumahdaf, A. (2016). Performance analysis of Markovian queue with impatience and vacation times. *ArXiv preprint arXiv*:1604.02449.

Choudhury, A., & Medhi, P. (2010). A Simple Analysis of Customers' Impatience in Multi-Server Queues. *Quantitative Methods*, 5, 182-198.

Gradshteyn, I., & Ryzhik, I. (2007). Table of Integrals, Series and Products. *Seventh edition, Elsevier Academic Press, Amsterdam.*

Gross, D., Shortle, J., Thompson, J., & Harris, C. (2017). *Fundamentals of Queuing Theory, Fifth Edition*. John Wiley and Sons, New York.

Haight, F.A. (1957). Queueing with balking. *Biometrika*, 47(3/4), 285-296.

Kumar, R., & Sharma, S.K. (2012). A Markovian feedback queue with retention of reneged customers. *Advanced modeling and optimization*, 14(3), 673-679.

Kumar, R., & Sharma, S.K. (2013). An M/M/c/N queueing system with reneging and retention of reneged customers. *International Journal of Operational Research*, 17(3), 333-334.

Laxmi, P.V., Kassahun, T.W., & Bhavani, E.G. (2019). Analysis of a Markovian queueing system with single working vacation and impatience of customers. *In Journal of Physics: Conference Series*, 1344(1), 012-018.

Lin, C.H., & Ke, J.C. (2009). Multi-server system with single working vacation. *Applied Mathematical Modeling*, 33(7), 2967-2977.

Liu, W., Xu, X., & Tian, N. (2007). Stochastic decompositions in the M/M/1 queue with working vacations. *Operations Research Letters*, 35, 595-600.

Lorentzen, L., & Waadeland, H. (2008). *Continued fractions, 2nd edition*. Atlantis Press, Amsterdam, Paris.

Majid, S., Manoharan, P., & Ashok, A. (2019). Analysis of an M/M/1 Queueing System with Working Vacation and Impatient Customers. *American International Journal of Researchin Science*, 2, 314-322.

Robert, E. (1979). Reneging phenomenon of single channel queues. *Mathematics of Operations Research*, 4(2), 162-178.

Servi, L., & Finn, S. (2002). M/M/1 queue with working vacations. *Performance Evaluation*, 50, 41-52.

Tian, N., Zhao, X., & Wang, K. (2008). The M/M/1 queue with a single working vacation. *International Journal of Information and Management Sciences*, 19(4), 621-634.

Tian, N., Li, J., Zhang, Z.G., & Luh, H.P. (2011). The M/G/1 queue with working vacations. *Queuing Systems*, 61, 139-166.

Yue, D., Yue, W., & Xu, G. (2012). Analysis of customer's impatience in an M/M/1 queue with working vacations. *Journal of Industrial and Management Optimization*, 8(4), 895-908.

Yue, D., Yue, W., Saer, Z., & Chen, X. (2014). Analysis of an M/M/1 queueing system with impatient customers and a variant of multiple vacation policy. *Journal of Industrial Management and Optimization*, 10(1), 89-112.

Yue, D., Yue, W., & Zhao, G. (2016). Analysis of an M/M/1 queue with vacations and impatience timers which depend on the server's states. *Journal of Industrial Management and Optimization*, 12(2), 653-666.