FULL-LENGTH ARTICLE

Approximate Analytical Solutions of Two-Dimensional Time Fractional Klein-Gordon Equation

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Abstract

This study presents a fractional reduced differential transform method (FRDTM) to find approximate analytical solutions of nonlinear time fractional two dimensional Klein Gordon equation. The fractional derivative used in this study is in the Caputo sense. A few important lemmas which are essential to solve the problems using the proposed method are proved. The advantage of this method is that it uses appropriate initial conditions and finds the solution to the problems without any discretization, linearization, perturbation, or any restrictive assumptions. Model examples are presented to demonstrate the applicability and effectiveness of the proposed method. The obtained results reveal that the FRDTM is very effective and simple for solving linear and non-linear fractional partial differential equations. Finally, some graphical features are presented to give a visual interpretation of the solutions behaviour.

Keywords: Klein-Gordon Equation; Caputo Fractional Derivative; Fractional Reduced Differential Transform Method

Introduction

In recent years, linear and nonlinear fractional partial differential equations have gained popularity and importance mainly, due their demonstrated applications in all disciplines of applied sciences and engineering, see (Ali et al., 2020a; Ali et al., 2020b; Arif et al., 2020; Sunthrayuth et al., 2021) and the references therein. The nonlinear Klein–Gordon equation (NLKGE) is a model of such a partial differential equation. This type of partial differential equation mainly arises in relativistic quantum mechanics and field theory, in high-energy physicists, in study of perturbation theory, in modelling the propagation of dislocations in crystals, spin waves, nonlinear optics, and in modelling the behaviour of elementary particles (Belayeh et al., 2020; Biswas et al., 2012; Chang and Kuo, 2014).

There have been attempts to develop new methods by different scholars to obtain approximate analytical solutions of fractional order differential equations which converge to exact solutions. A few of these schemes are the Adomain decomposition method (ADM) (Daftardar-Gejji and Jafari, 2005; Daftardar-Gejji and Jafari, 2007; Duan et al., 2012), the differential transform method (DTM) (Ghazanfari and Ebrahim, 2015; Secer et al., 2012), the reduced differential transform method (RDTM) (Deresse et al., 2021; Jafari et al., 2016). The majority of these methods are sometimes complex and contains terms not easily calculable, and it is difficult to obtain approximate analytical solutions. To deal with such types of difficulties, Keskin and Oturanc, (2010) have developed an alternative method, known as fractional reduced differential transform method (FRDTM). Very recently, in papers (Arshad and Lu, 2017; Mussa et al., 2021; Srivastava et al., 2014; Taghavi et al., 2015; Yadeta et al., 2020) different scholars have shown the capabilities of FRDTM in solving linear and nonlinear fractional order partial differential equations .

Bearing in mind the capability of FRDTM, the central goal of this paper is to find approximate analytical solutions of time fractional NLKGE in two dimensions of the form:

$$\frac{\partial^{2\alpha}u(x,y,t)}{\partial t^{2\alpha}} + \beta\frac{\partial^{\alpha}u(x,y,t)}{\partial t^{\alpha}} = \theta\left(\frac{\partial^{2}u(x,y,t)}{\partial x^{2}} + \frac{\partial^{2}u(x,y,t)}{\partial y^{2}}\right) - Nu + f(x,y,t) \qquad (1)$$

subject to the initial conditions

$$u(x,y,0) = \varphi_1(x,y) \text{ and } u_t(x,y,0) = \varphi_2(x,y) , \qquad (2)$$

where $t > 0$, $0 < \alpha \leq 1$, Nu is a nonlinear term, φ_1 and φ_2 are prescribed functions in two space variables, β is the so-called dissipative term, which is assumed to be a real number with $\beta \geq 0$. When $\beta = 0$, Equation (1) reduces to the undamped Klein–Gordon equation, while when $\beta > 0$, to the damped one, and θ is non negative real number.

Materials and Method

Preliminaries

We consider first the following basic definitions that can be used for the next sections.

Definition 1. The gamma function. *The gamma function $\Gamma(z)$ is simply a generalization of the factorial real arguments. The Gamma function can be defined as* (Podlubny, 1999)

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}dt,\ z \in \mathbb{C} \qquad (3)$$

Definition 2. *The Caputo fractional derivative is defined as* (Podlubny, 1999)

$$D_t^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{n}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau , & n-1 < \alpha < n \\ \frac{d^n}{dt^n} f(t) , & \alpha = n \end{cases}, \qquad (4)$$

where $\alpha > 0,\ t > a,\ n \in N,$ and $\alpha,\ a,\ t \in$ R.

The Caputo fractional derivative is one among the several definitions of fractional derivatives. We use this definition in this paper for the reason that the initial conditions of the fractional order differential equation are in a form involving only the limit values of integer-order derivative at the lower terminal initial time ($t = a$), and also the fractional derivative of a constant function is zero.

i) $D_a^{\alpha}\mathrm{C} = 0$, C is a constant.

ii) $D_a^{\alpha}\mathrm{t}^{\gamma} = \begin{cases} 0 & , \quad \gamma \leq \alpha - 1 \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha} , & \gamma > \alpha - 1 \end{cases}$

Similar to integer order differentiation, fractional order differentiation in Caputo's sense is a linear operation ,

$$D_a^{\alpha}(\xi f(x) + \eta g(x)) = \xi D_a^{\alpha} f(x) + \eta D_a^{\alpha} g(x) , \qquad (5)$$

where $\xi$and η are constants.

Fractional Reduced Differential Transform Method (FRDTM)

The following definitions and properties of FRDTM are according to the references (Arshad and Lu, 2017; Srivastava et al., 2014; Taghavi et al., 2015).

Definition 3. If $u(x,y,t)$ is analytic and continuously differentiable with respect to space variables x,y and time variable t in the domain of interest, then the fractional reduced differential transform of the function $u(x,y,t)$ is defined as

$$U_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}u(x,y,t)}{\partial t^{k\alpha}}\right]_{t=t_0}, \text{ where } k = 0,1,2, \ . \ . \ . \qquad (6)$$

Definition 4. The fractional reduced differential inverse transform of $U_k(x,y)$ is defined as

$$R_D^{-1}[U_k(x,y)] \cong u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)(t-t_0)^{k\alpha} \qquad (7)$$

From Eqs. (6) and (7) one can deduce that

$$u(x,y,t) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}u(x,y,t)}{\partial t^{k\alpha}}\right]_{t=t_0}(t-t_0)^{k\alpha} \qquad (8)$$

Note that when $t_0 = 0$, Eq. (7) becomes

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^{k\alpha} \qquad (9)$$

From Definition 4, it can be seen that the concept of FRDTM is derived from the power series expansion of a function. Then, the inverse transform of the set of values $[U_k(x,y)]_{k=0}^{n}$ gives the n-term approximate solutions:

$$\bar{u}_n(x,y,t) = \sum_{k=0}^{n} U_k(x,y)\, t^{\alpha k} \qquad (10)$$

Therefore, the exact solution is given by:

$$u(x,y,t) = \lim_{n\to\infty}\bar{u}_n(x,y,t) = \sum_{k=0}^{\infty} \mathrm{U}_k(x,y)\mathrm{t}^{\alpha k} \qquad (11)$$

Some of the basic properties of FRDTM that can be deduced from Eqs. (6) and (7) are given below, where $k = 0,1,2,\cdots$, and α, β and γ are constants.

1. If $w(x,y,t) = \alpha u(x,y,t) + \beta v(x,y,t)$, then $W_k(x,y) = \alpha U_k(x,y) + \beta V_k(x,y)$,
2. If $w(x,y,t) = u(x,y,t)v(x,y,t)$, then $W_k(x,y) = \sum_{r=0}^{k} U_r(x,y)V_{k-r}(x,y)$
3. If $w(x,y,t) = u(x,y,t)v(x,y,t)s(x,y,t)$, then
$W_k(x,y) = \sum_{j=0}^{k}\sum_{i=0}^{j} V_i(x,y)U_{j-i}(x,y)S_{k-j}(x,y)$
4. If $w(x,y,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,y,t)$, then $W_k(x,y) = \frac{\Gamma((k+N)\alpha+1)}{\Gamma(k\alpha+1)}U_{k+r}(x,y)$
5. If $w(x,y,t) = \frac{\partial^2}{\partial x^2}u(x,y,t)$, then $W_k(x,y) = \frac{\partial^2}{\partial x^2}U_k(x,y)$
6. If $w(x,y,t) = \frac{\partial^2}{\partial y^2}u(x,y,t)$, then $W_k(x,y) = \frac{\partial^2}{\partial y^2}U_k(x,y)$
7. If $w(x,y,t) = sin(\alpha x + \gamma y + \beta t)$, then $W_k(x,y) = \frac{\beta^k}{k!}sin\left(\alpha x + \gamma y + \frac{k\pi}{2}\right)$,
8. If $w(x,y,t) = cos(\alpha x + \gamma y + \beta t)$, then $W_k(x,y) = \frac{\beta^k}{k!}cos\left(\alpha x + \gamma y + \frac{k\pi}{2}\right)$.

Lemma 1 If $w(x,y,t) = sin(\omega y)\, sin(\alpha x + \beta t)$, then $W_k(x,y) = \frac{\beta^k}{k!} sin\left(\alpha x + \frac{k\pi}{2}\right) sin(\omega y)$,

where $\omega,\ \alpha,\ \beta$ are constants and $k = 0, 1, 2, 3,\ \ldots\ ,$ (12)

Proof: By Definition 3 and basic properties of FRDTM, we have

$$W_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}(sin(\omega y) sin(\alpha x+\beta t))}{\partial t^{k\alpha}}\right]_{t=t_0},$$
$$W_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha} sin(\alpha x+\beta t)}{\partial t^{k\alpha}}\right]_{t=t_0} sin(\omega y),$$
$$W_k(x,y) = \frac{\beta^k}{k!} sin\left(\alpha x + \frac{k\pi}{2}\right) sin(\omega y),$$

where $\omega, \alpha,\ \beta$ are constants and $k = 0, 1, 2, 3,\ \ldots,$.

Implementation of the Method

To demonstrate the basic concepts of the FRDTM, we consider the time fractional NLKGE in two dimensions given in Eq. (1) with initial conditions (2). According to Definition 3 and basic properties of FRDTM discussed in the previous section, we can construct the following iteration formula:

$$\frac{\Gamma((k+2)\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}(x,y) + \beta \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x,y) = \theta\left[\frac{\partial^2 U_k(x,y)}{\partial x^2} + \frac{\partial^2 U_k(x,y)}{\partial y^2}\right] - N_k(x,y) + F_k(x,y), \quad (13)$$

where $N_k = N\big(U_k(x,y)\big)$ and $F_k(x,y)$ are the transformed form of the nonlinear term Nu and the source function $f(x,y,t)$ respectively. Similarly, the transformed form the initial conditions (2) are

$$U_0(x,y) = \varphi_1(x,y) \text{ and } U_1(x,y) = \varphi_1(x,y) \text{ respectively.} \quad (14)$$

The first nonlinear terms are computed below,

$$\left.\begin{aligned} N_0 &= N(U_0) \\ N_1 &= N^{(1)}(U_0).\, U_1 \\ N_2 &= N^{(1)}(U_0).\, U_2 + \frac{1}{2!} N^{(2)}(U_0).\, U_1^2 \\ N_3 &= N^{(1)}(U_0).\, U_3 + N^{(2)}(U_0).\, U_1.\, U_2 + \frac{1}{3!} N^{(3)}(U_0).\, U_1^3 \end{aligned}\right\}, \quad (15)$$

where $N^{(k)}(U_0)$ is the k^{th} order derivative of the nonlinear term at U_0.
Substituting the initial conditions (4.4) and the components of $N\big(U_k(x,y)\big)$ into (13) and straight forward iteration, we obtain the values of $\{U_k(x,y)\}_{k=0}^{\infty}$. Then, the inverse fractional reduced differential transform can be obtained as

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y) t^k$$
$$= U_0(x,y) + U_1(x,y)t^1 + U_2(x,y)t^2 + U_3(x,y)t^3 + U_4(x,y)t^4 + \cdots \quad (16)$$

Convergence of the Method

We study the convergence of the FRDTM when it is used in Equations (1) and (2).

Theorem 1. The series solution $\sum_{k=0}^{\infty} U_k(x,y)t^k$ given in Eq. (16), converges if $\exists\ 0 < \lambda < 1$ such that

$$||U_{k+1}(x,y)|| \le \lambda ||U_k(x,y)||,\ \forall k \in N \cup \{0\} \quad (17)$$

Proof:
Let $(C[l], ||.||)$ be the Banach space of all continuous function on l with the norm $||U_k(x,y)||$. Also, assume that $||U_0(x,y)|| < \eta_0$, where η_0 is a positive number. Define the sequence of partial sum $\{S_n\}_{n=0}^{\infty}$ as

$$S_n = U_0 + U_1 + U_2 + \dots + U_n \quad (18)$$

We want to show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in this Banach space. To achieve this goal,
we take

$$||S_{n+1} - S_n|| = ||U_{n+1}|| \leq \lambda||U_n|| \leq \lambda^2||U_{n-1}|| \leq \dots \leq \lambda^{n+1}||U_0|| \leq \lambda^{n+1}\eta_0 \quad (19)$$

For any $n,\ m \in N, n \geq m$, we have

$$\begin{aligned} ||S_n - S_m|| &= ||(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \dots + (S_{m+1} - S_m)|| \\ &\leq ||(S_n - S_{n-1})|| + ||(S_{n-1} - S_{n-2})|| + \dots + ||(S_{m+1} - S_m)|| \\ &\leq \lambda^n||U_0|| + \lambda^{n-1}||U_0|| + \dots + \lambda^{m+1}||U_0|| \\ &\leq (\lambda^{n-m-1} + \lambda^{n-m-2} + \dots + 1)||U_0|| \\ &\leq \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right)\lambda^{m+1}||U_0||, \end{aligned} \quad (20)$$

and because $0 < \lambda < 1$, we obtain

$$lim_{n,m\to\infty} ||S_n - S_m|| = 0$$

(21)

Therefore, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[l], ||.||)$. Then the series solution $\sum_{k=0}^{\infty} U_k(x,y)t^k$, defined in Eq. (x1) converges and completes the proof.

If the series solution $\sum_{k=0}^{\infty} U_k(x,y)t^k$ convergence then it is an exact solution of time fractional NLKGE in two dimensions given in Eq. (1).

Theorem 2. Suppose that the series solution $\sum_{k=0}^{\infty} U_k(x,y)t^k$ converges to the solution $u(x,y,t)$.
If the truncated series $\sum_{k=0}^{m} U_k(x,y)t^k$ is used as an approximation to the solution $u(x,y,t)$, then
the maximum absolute truncated error is computed as

$$||u(x,y,t) - \sum_{k=0}^{m} U_k(x,y)t^k|| \leq \left(\frac{1}{1-\lambda}\right)\lambda^{m+1}||U_0||, \quad (22)$$

Proof.
According to Theorem (1), by following the inequality Eq. (20), for $n \geq m$, we have

$$||S_n - S_m|| \leq \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right)\lambda^{m+1}||U_0||, \quad (23)$$

Also, since $0 < \lambda < 1$, we get $1 - \lambda^{n-m} < 1$, therefore, the inequality (23) can be changed to

$$||S_n - S_m|| \leq \left(\frac{1}{1-\lambda}\right)\lambda^{m+1}||U_0|| \quad (24)$$

It is evident when $n \to \infty$, $S_n \to u(x,y,t)$. Thus, Eq. (22) is obtained. This completes the proof.

Model examples and Results
To demonstrate the performance and efficiency of the proposed method two model examples are illustrated.

Example 1. Consider the following time fractional NLKGE

$$\frac{\partial^{2\alpha}u(x,y,t)}{\partial t^{2\alpha}} - \frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{\partial^2 u(x,y,t)}{\partial y^2} + u^2(x,y,t) = -xy\, cos(t) + x^2y^2cos^2(t), t > 0, \quad (25)$$

with initial conditions,

$$u(x,y,0) = xy \text{ and } u_t(x,y,0) = 0, \quad (26)$$

where $0 < \alpha \leq 1$ is the order of the fractional derivative in the sense of Caputo.

Applying properties of FRDTM to Eq. (25), we obtain the following recurrence relation

$$U_{k+2}(x,y) = \frac{\Gamma(k\alpha+1)}{\Gamma((k+2)\alpha+1)}\left[\frac{\partial^2 U_k(x,y)}{\partial x^2} + \frac{\partial^2 U_k(x,y)}{\partial y^2} - N_k(x,y) + F_k(x,y)\right]. \quad (27)$$

where $N_k(x,y)$ is the fractional reduced differential transform of $u^2(x,y,t)$ and $F_k(x,y)$ is the fractional reduced differential transform of the function $f(x,y,t) = -xy\cos(t) + x^2y^2\cos^2(t)$.

By similar scheme from (26), we obtain

$$U_0(x,y) = xy \text{ and } U_1(x,y) = 0. \quad (28)$$

Let

$$g(x,y,t) =- xy\cos(t) \quad (29)$$

and

$$h(x,y,t) = g^2(x,y,t) = x^2y^2\cos^2(t). \quad (30)$$

Applying properties of FRDTM on Eqs. (29) and (30), we get

$$G_k(x,y) = -xy\frac{1}{k!}\cos\left(k\frac{\pi}{2}\right), k = 0, 1, 2, 3, \; . . . \quad (31)$$

and

$$H_k(x,y) = \sum_{i=0}^{k} G_i(x,y)G_{k-i}(x,y), \; k = 0, 1, 2, 3, \; . . . \quad (32)$$

respectively.

Thus,

$$F_k(x,y) = G_k(x,y) + H_k(x,y). \quad (33)$$

We then compute the set of values of $\{u_k(x,y)\}_{k=0}^{\infty}$.

When $k = 0$, from Eqs. (15), (31) and (32), we obtain

$$N_0(x,y) = U_0^2(x,y) = x^2y^2,$$
$$G_0(x,y) = -xy \text{ and } \quad (34)$$
$$H_0(x,y) = \sum_{i=0}^{0} G_i(x,y)\, G_{k-i}(x,y) = G_0^2(x,y) = x^2y^2$$

respectively.

So, equation (33) when $k = 0$ yields

$$F_0(x,y) = G_0(x,y) + H_0(x,y) = -xy + x^2y^2. \quad (35)$$

Hence, from Eq. (27) when $k = 0$, we obtain

$$U_2(x,y) = \frac{1}{\Gamma(2\alpha+1)}\left[\frac{\partial^2 U_0(x,y)}{\partial x^2} + \frac{\partial^2 U_0(x,y)}{\partial y^2} - N_0(x,y) + F_0(x,y)\right]$$
$$= -\frac{xy}{\Gamma(2\alpha+1)} \quad (36)$$

Accordingly,

$$N_1(x,y) = 2U_1(x,y)U_0(x,y) = 0,$$
$$G_1(x,y) = 0, \quad (37)$$
$$H_1(x,y) = \sum_{i=0}^{1} G_i(x,y)\, G_{k-i}(x,y) = 0$$

We have, $F_1(x,y) = G_1(x,y) + H_1(x,y) = 0$ and hence, when $k = 1$, Eq. (27) yields

$$U_3(x,y) = \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left[\frac{\partial^2 U_1(x,y)}{\partial x^2} + \frac{\partial^2 U_1(x,y)}{\partial y^2} - N_1(x,y) + F_1(x,y)\right] = 0, \quad (38)$$

Using similar procedure we obtain the following results

$$U_4(x,y) = \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(x^2y^2\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right) + \frac{xy}{2}\right),$$
$$U_5(x,y) = 0,$$
$$U_6(x,y) = \frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\left[\begin{array}{l}\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(2(x^2+y^2)\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right)\right) \\ -\left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}(2x^3y^3)\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right) + x^2y^2\right) + \frac{x^2y^2}{3} - \frac{xy}{4!}\end{array}\right] \quad (39)$$

and so on.

The fractional reduced differential inverse transform of $\{U_k\}_{k=0}^{\infty}$ gives the following series solution:

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^{k\alpha}$$
$$= U_0(x,y) + U_1(x,y)t^{\alpha} + U_2(x,y)t^{2\alpha} + U_3(x,y)t^{3\alpha} + U_4(x,y)t^{4\alpha} + \cdots$$

$$u(x,y,t) = \left[\begin{array}{l} xy - \frac{xy}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(x^2y^2\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right) + \frac{xy}{2}\right)t^{4\alpha} \\ + \frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\left[\begin{array}{l}\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(2(x^2+y^2)\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right)\right) \\ -\left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}(2x^3y^3)\left(\frac{2}{\Gamma(2\alpha+1)} - 1\right) + x^2y^2\right) + \frac{x^2y^2}{3} - \frac{xy}{4!}\end{array}\right]t^{6\alpha} + \cdots \end{array}\right], \quad (40)$$

In particular, if in Equation (40) $\alpha = 1$, we obtain

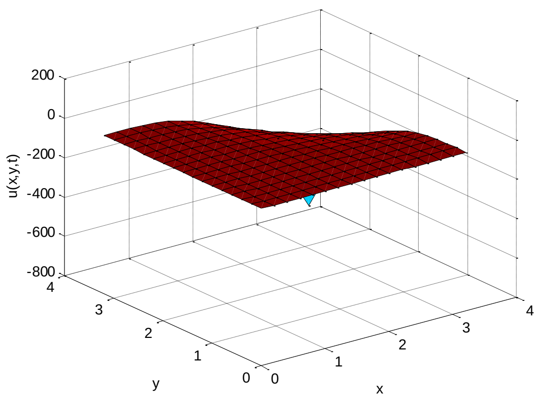
$$u(x,y,t) = xy\left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots\right). \quad (41)$$

which converges efficiently to the exact solution $u(x,y,t) = xy\cos(t)$, (Belayeh et al., 2020).

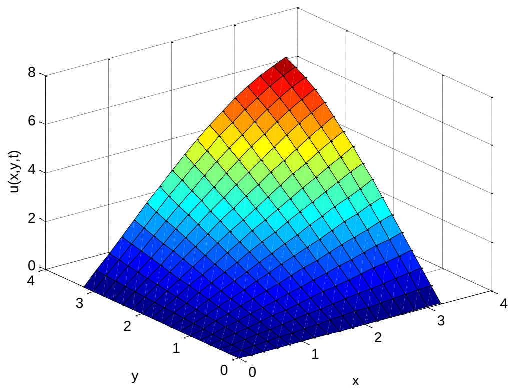
Numerical results corresponding to Example 1 are depicted in table 1 and figure 1.

Table 1. Approximate solution of Example 1 for different values of fractional order α at $x = y = 1$, and comparison with the exact solution.

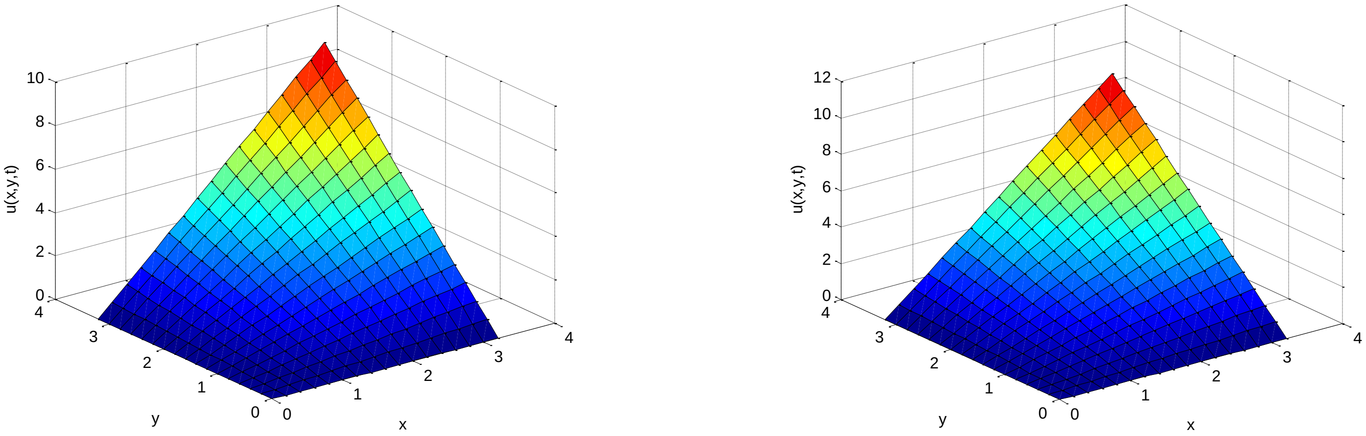
t	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$	Exact	Absolute Error $\lvert u_{Exact} - u_{\alpha=1} \rvert$
0.1	0.856759092	0.944652453	0.982533952	0.995004165	0.995004165	2.48E-13
0.2	0.785269046	0.878448085	0.947688257	0.980066578	0.980066578	6.35E-11
0.3	0.746078516	0.812139523	0.901550971	0.955336488	0.955336489	1.63E-09
0.4	0.730951748	0.749195265	0.847103358	0.921060978	0.921060994	1.62E-08
0.5	0.735767941	0.691541894	0.786522413	0.877582465	0.877582562	9.66E-08
0.6	0.757971505	0.640381058	0.721578900	0.825335200	0.825335615	4.15E-07
0.7	0.795789515	0.596477428	0.653770474	0.764840765	0.764842187	1.42E-06
0.8	0.847903105	0.560293526	0.584374095	0.696702578	0.696706709	4.13E-06
0.9	0.913283179	0.532063951	0.514466126	0.621599388	0.621609968	1.06E-05
1	0.991098338	0.511840836	0.444927334	0.540277778	0.540302306	2.45E-05



(a)



(b)



(c) (d)

Figure 1. 3 D view of the solution behaviour of Example 1 at $t = 0.1$ when (a) $\alpha = 0.1$, (b) $\alpha = 0.4$, (c) $\alpha = 0.7$, and (d) $\alpha = 1$.

Example 2. Consider the following time fractional NLKGE

$$\frac{\partial^{2\alpha}u(x,y,t)}{\partial t^{2\alpha}}-\frac{\partial^2 u(x,y,t)}{\partial x^2}-\frac{\partial^2 u(x,y,t)}{\partial y^2}+u^3(x,y,t)=\cos(x)\cos(y)\sin(t)+\cos^3(x)\cos^3(y),$$
$$t>0,\ \ 0<\alpha\leq 1$$
(42)

with initial conditions,

$$u(x,y,0)=0 \ \text{ and } \ u_t(x,y,0)=\cos(x)\cos(y). \qquad (43)$$

By applying FRDTM on Eq. (42), the following recursive equation is obtained:

$$U_{k+2}(x,y)=\frac{\Gamma(\mathrm{k}\alpha+1)}{\Gamma((\mathrm{k}+2)\alpha+1)}\left[\frac{\partial^2 U_k(x,y)}{\partial x^2}+\frac{\partial^2 U_k(x,y)}{\partial y^2}-N_k(x,y)+F_k(x,y)\right] \qquad (44)$$

where $N_k(x,y)$ is the transformed form of $u^3\,(x,y,t)$ and $F_k(x,y)$ is the transformed form of the function $f\,(x,y,t)=\cos(x)\cos(y)\sin(\,t)+\cos^3(x)\cos^3(y)\sin^3(\,t)$.
The fractional reduced differential transform of Eq. (43) is

$$U_0(x,y)=0 \ \text{ and } \ U_1(x,y)=cos(x)\,cos(y). \qquad (45)$$

Let $g(x,y,t)=cos(x)\,cos(y)\,sin(\,t)$
$$=\frac{1}{4}[sin(t+x-y)+sin(t-x+y)+sin(t+x+y)+sin(t-x-y)]$$
(46)
and

$$h(x,y,t)=g^3(x,y,t)=cos^3(x)\,cos^3(y)\,sin^3(t). \qquad (47)$$

Then the fractional reduced differential transform of the Eqs. (46) and (47) are

$$G_k(x,y)=\frac{1}{4}\left[\frac{1}{k!}sin\left(k\frac{\pi}{2}+x-y\right)+\frac{1}{k!}sin\left(k\frac{\pi}{2}-x+y\right)+\frac{1}{k!}sin\left(k\frac{\pi}{2}+x+y\right)+\frac{1}{k!}sin\left(k\frac{\pi}{2}-x-y\right)\right] \qquad (48)$$

and

$$H_k(x,y)=\ \sum_{j=0}^{k}\sum_{i=0}^{j}G_i(x,y)\,G_{j-i}(x,y)G_{k-j}(x,y) \qquad (49)$$

respectively. Therefore,

$F_k\,(x,y)=G_k\,(x,y)+H_k\,(x,y)$ as defined in Eq. (33).

Next we compute the set of values of $\{u_k(x,y)\}_{k=0}^{\infty}$.
From Eqs. (15), (48) and (49) when $k=0$, we obtain

$$N_0(x,y)=\sum_{j=0}^{0}\left(\sum_{i=0}^{j}U_i\,U_{j-i}U_{k-j}\right)=U_0^3(x,y)=0$$
$$G_0(x,y)=\frac{1}{4}[sin(x-y)+sin(-x+y)+sin(x+y)+sin(-x-y)]=0,\ and \qquad (50)$$
$$H_0(x,y)=\sum_{j=0}^{0}\sum_{i=0}^{0}G_i(x,y)\,G_{j-i}(x,y)G_{k-j}(x,y)=G_0^3(x,y)=0$$

respectively.

Thus, $F_0(x,y)=G_0(x,y)+H_0(x,y)=0$, and therefore, Eq. (44) when $k=0$ becomes

$$U_2(x,y)=\frac{1}{\Gamma(2\alpha+1)}\left[\frac{\partial^2 U_0(x,y)}{\partial x^2}+\frac{\partial^2 U_0(x,y)}{\partial y^2}-N_0(x,y)+F_0(x,y)\right]=0 \qquad (51)$$

When $k=1$, we have

$N_1(x,y) = 3U_0^2(x,y)U_1(x,y) = 0$

$G_1(x,y) = \frac{1}{8}\left[sin\left(\frac{\pi}{2} + x - y\right) + sin\left(\frac{\pi}{2} - (x - y)\right) + sin\left(\frac{\pi}{2} + x + y\right) + sin\left(\frac{\pi}{2} - (x + y)\right)\right],$

$= cos\, x\, cos\, y,$

$H_1(x,y) = \sum_{j=0}^{1}\sum_{i=0}^{j} G_i(x,y)\, G_{j-i}(x,y)G_{k-j}(x,y) = 3G_0^2(x,y)G_1(x,y) = 0$

(52)

As a result, $F_1(x,y) = G_1(x,y) + H_1(x,y) = cos(x)\, cos(y)$. So, Eq. (44) implies

$$U_3(x,y) = \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left[\frac{\partial^2 U_1(x,y)}{\partial x^2} + \frac{\partial^2 U_1(x,y)}{\partial y^2} - N_1(x,y) + F_1(x,y)\right] = -\frac{cos(x)\, cos(y)\Gamma(1+\alpha)}{\Gamma(1+3\alpha)} \qquad (53)$$

In a similar manner, we obtain

$$U_4(x,y) = 0\,,$$
$$U_5(x,y) = \frac{cos(x)cos(y)(12\,\Gamma(1+\alpha) - \Gamma(1+3\alpha))}{6\Gamma(1+5\alpha)} \qquad (54)$$
$$U_6(x,y) = 0$$

and so on.

The differential inverse transform of $\{U_k\}_{k=0}^{\infty}$ will give the following series solution:

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^{k\alpha}$$

$$u(x,y,t) = cos\, x\, cos\, y\, t^{\alpha} - \frac{cos(x)\, cos(y)\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\, t^{3\alpha} + \frac{cos(x)\, cos(y)\big(12\,\Gamma(1+\alpha) - \Gamma(1+3\alpha)\big)}{6\Gamma(1+5\alpha)} t^{5\alpha} - \ldots$$

(55)

In particular, if $\alpha = 1$, in Eq. (55) becomes

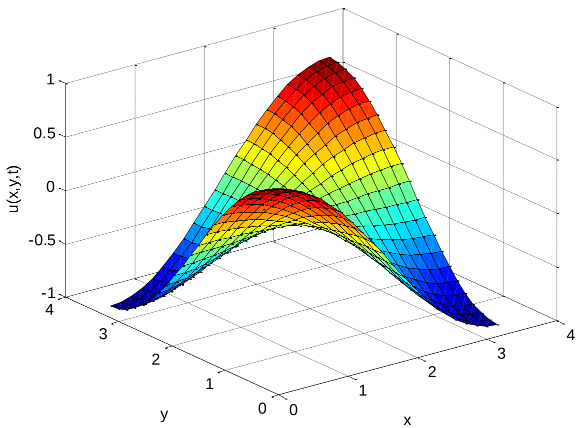
$$u(x,y,t) = cos(x)\, cos(y)\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\right). \qquad (56\,)$$

The exact solution of Example 2 is $u(x,y,t) = \cos(x)\cos(y)\sin(t)$ as in (Chang and Kuo, 2014).

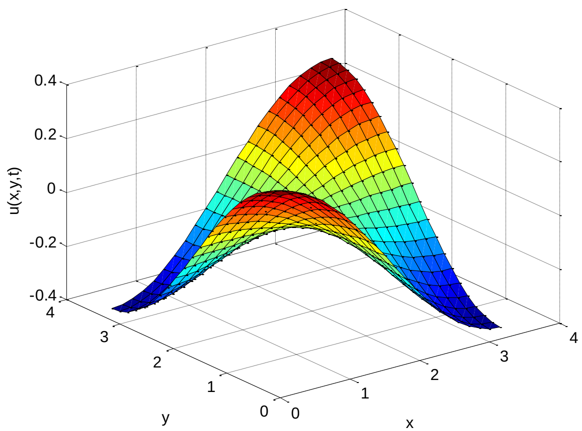
Numerical results corresponding to the two dimensional NLKGE given in Example 2 are depicted in table 2 and figure 2.

Table 2. Approximate solution of Example 2 for different values of fractional order α at $x = y = 1$, and comparison with the exact solution.

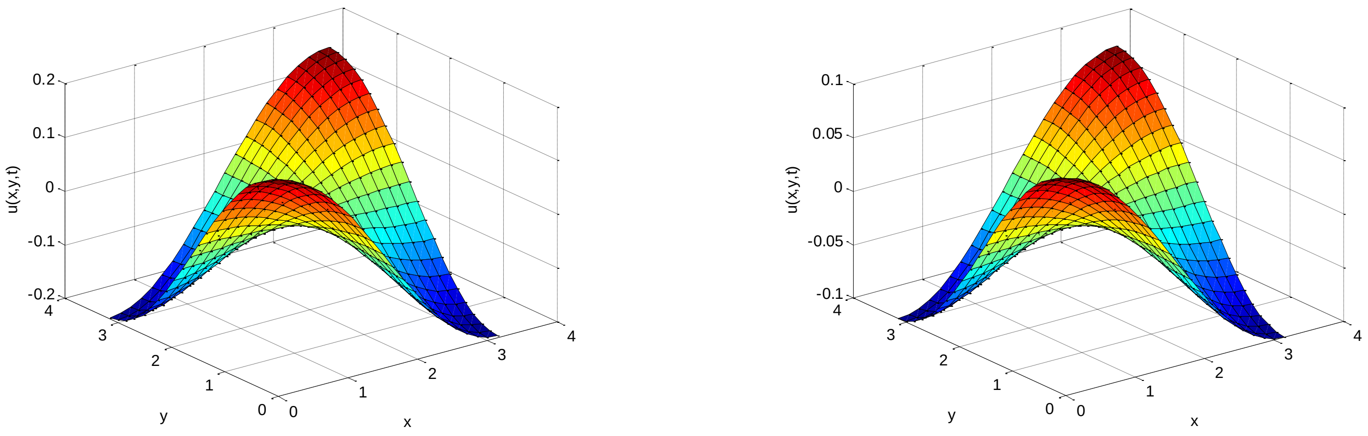
t	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$	Exact	Absolute Error $\lvert u_{Exact} - u_{\alpha=1} \rvert$
0.1	0.103707403	0.070936099	0.045905820	0.029144028	0.029144028	5.79E-12
0.2	0.128562353	0.103145512	0.078666153	0.057996859	0.057996859	7.40E-10
0.3	0.145818607	0.125922839	0.106485408	0.086270216	0.086270204	1.27E-08
0.4	0.161213440	0.143258743	0.130566657	0.113681660	0.113681566	9.47E-08
0.5	0.176965847	0.157087985	0.151426402	0.139957510	0.139957059	4.51E-07
0.6	0.194221367	0.168672013	0.169384673	0.164835761	0.164834147	1.61E-06
0.7	0.213667235	0.178970147	0.184699751	0.188069005	0.188064267	4.74E-06
0.8	0.235756382	0.188780924	0.197619331	0.209427351	0.209415311	1.20E-05
0.9	0.260807313	0.198807392	0.208403716	0.228701342	0.228673947	2.74E-05
1	0.28905491	0.209691335	0.217337613	0.245704873	0.245647748	5.71E-05



(a)



(b)



(c) (d)

Figure 2. 3 D view of the solution behaviour of Example 2 at $t = 0.1$ when (a) $\alpha = 0.1$, (b) $\alpha = 0.4$, (c) $\alpha = 0.7$, and (d) $\alpha = 1$.

Discussion

Tables 1 and 2 exhibit the solutions behaviour of Examples 1 and 2, respectively for different values of fractional order α and t, and comparisons of the approximated solutions by the proposed method with the corresponding exact solutions. As it can be seen in tables 1 and 2 the proposed method is in good agreement with the exact solutions under suitable initial conditions when $0 < t < 1$. Further, when α approaches to 1, the corresponding approximate solutions are closer and closer to the exact solutions. Thus, the proposed method yields good approximate solutions for small values of t and for α close to 1, whatever the values of x and y are within the domain of interest.

Figures 1 and 2 demonstrate the physical behaviour of the solution graphs of Examples 1 and 2 for different values of fractional order α, and also these figures depicts that when α goes to 1 the approximated solution graphs resemble the graph of the corresponding exact solution of the classical two dimensional NLKGE, (Belayeh et al. 2020). Further, solution depends on the time-fractional derivative. Accuracy and efficiency can be enhanced by increasing the number of iterations.

Conclusion

In this paper, FRDTM is implemented for solving time fractional two-dimensional NLKGE. The scheme gives a series solution which converges rapidly to exact or approximate solutions with easily calculable terms. Applicability of the method is investigated by considering two model examples examples; only small amounts of computations gave rapid convergence to the exact solutions, and only a few iterations are enough to yield good approximate solutions. Effects of the fractional order α and t on the approximate solutions are shown using tables 1 and 2, and figures 1 and 2. The results obtained reveal that FRDTM is a reliable and powerful method for solving different types of higher orders nonlinear time fractional partial differential equations.

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